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ABSTRACT
This paper proposes a novel S-parameter retrieval method to extract the arbitrary permittivity and permeability tensors of general biaxial anisotropic materials. The proposed method uses three measurements of the reflection and transmission coefficients of a simple normal incident circularly polarized plane wave. The circular polarization is employed to overcome the ambiguity attributed to the discontinuities in the extracted possible positive and negative values of the permittivity and permeability tensors for natural or synthesized anisotropic materials. The mathematical procedure of the proposed method directly relates the combined parameters deduced from the circularly polarized wave components to the tensor permittivity and permeability linear components while avoiding the need for heavy computations or optimization schemes. This simple procedure successfully overcomes the limitations to the usable frequency band and the material losses observed in previously published retrieval techniques for lossless and lossy anisotropic materials at low- and high-frequency regions. The retrieved tensor elements are validated by comparison to analytical data obtained from solving the original electromagnetic problem using the Berreman 4 × 4 matrix method after extending it to the case of circularly polarized fields.

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I. INTRODUCTION
The fast developments in contemporary material technology are accompanied by extensive work to understand the complex characterization at terahertz (THz) frequencies of materials of interest, facilitating effective designs and applications. Several techniques for extracting scalar and equivalent constitutive parameters have been used to characterize complex electromagnetic materials.1–9 Such extraction methods proceed by measuring transmission and reflection free space S-parameters from the material specimens. The retrieval technique proposed by Nicolson and Ross10 and that proposed by Smith et al.7,11 are used to extract the effective scalar permittivities and permeabilities for artificial and natural materials. However, these methods are prone to obtaining ambiguous permeability and permittivity values, given the logarithmic nature of the real part function of the material refractive index.11,12

Anisotropic materials comprise a class of complex materials, the properties of which are direction-dependent and described by tensors instead of scalar structural parameters. Recent studies on the characterization of anisotropic and simplified bianisotropic materials have attracted much interest. For achieving a simplified bianisotropic metamaterial, Marques et al.13 presented an equivalent material tensor model for a split-ring resonator (SRR) structure. In Refs. 14–18, the SRR metamaterial is considered reciprocal, and it is described by tensor-equivalent constitutive material parameters. Chen et al. presented a retrieval technique from the S-parameters using three normal wave incidence directions and six different measurements along with an optimization approach for the determination of the unknown equivalent parameters of an SRR periodic structure.15 In Ref. 19, two optimization methods to minimize the relative parameter mismatch are applied to the retrieval of the constitutive tensors of the SRR structure using three obliquely incident plane wave measurements. Analytical inversion equations are derived in Ref. 18 to retrieve the effective parameter tensors of an SRR cell using four different S-parameters of normal-incident waves. A similar procedure using the transfer-matrix method is used in Ref. 16. In Ref. 20, the S-parameters of a single SRR cell are measured using two orthogonal polarizations (TE and TM) at each of the two oblique positions and one normal position. Characterization of the lossy
An analytical technique built upon the $4 \times 4$ matrix method was introduced in Ref. 22 to extract arbitrary permittivity and permeability tensors of lossy anisotropic materials using multiple TE and TM off-axis oblique measurements in addition to an optimization scheme. Efforts for anisotropic material characterization of SRR and similar structures are based on data obtained from numerical simulations, generally lacking appropriate analytical result validation. In addition, the methods that use optimization schemes are usually accompanied by heavy computation cost and often yield multiple solutions. Nonetheless, the major downside of the existing methods for anisotropy characterization is their limitations regarding lossless materials where the problem of multiple branching ambiguity occurs at low- and high-frequency regions. At high frequencies, even with lossy materials, the multiple branching ambiguity appears, which limits such methods at high frequencies, especially for thick material slabs. Phase unwrapping techniques have been applied efficiently in correcting errors in the computed phase due to the multi-branching discontinuity problem in several structures.\footnote{20,26,27} In general, phase unwrapping will be limited by the fact that the phase of the refractive index is sensitive to small ripples in the S-parameter measurements, especially for thicker slabs.\footnote{28} In phase unwrapping, the starting point must be determined precisely in magnitude and phase before initializing the procedure due to the accumulative nature of the branching errors. In the case of complex anisotropic materials, there is a need to determine the starting frequency that has zero branching for all the tensor elements.\footnote{22} If the starting point is correct, the solution will be multiple again after frequencies with zero refractive index. In addition, when the desirable frequency band extends to high frequencies, the phase unwrapping technique becomes computationally inefficient due to the need for computations at lower starting regions.\footnote{26}

This paper presents a novel tensor parameter retrieval method at THz frequencies based on circular wave incidence for homogeneous optical/electromagnetic anisotropic structures. This method is simple, requiring only three different measurements. It also overcomes the aforementioned material losses and frequency limitations suffered by existing methods. The $4 \times 4$ matrix method is analytically extended to treat circularly polarized fields and solve for the reflection and transmission S-parameters for the lossless biaxial anisotropic material. The circular polarization is employed to overcome the discontinuities in the extracted possible positive and negative values of the permittivity and permeability tensors for natural or synthesized anisotropic materials. The results of the retrieved parameters are validated by comparison to the original analytical data, and the absolute error in the permittivity and permeability tensors is computed.

II. THEORETICAL ANALYSIS

The Berreman $4 \times 4$ matrix method\footnote{29} is one of the most general, robust, and implementable means of analyzing electromagnetic wave propagation in arbitrarily anisotropic multilayer structures. Here, the method has been extended to solve for the transmission and reflection coefficients of circular polarization field components in a stratified anisotropic material layer in free space. The Berreman $4 \times 4$ matrix representation reduces the Maxwell equations to four ordinary differential equations for the tangential components of the electromagnetic fields ($E, H$) as follows:

$$
\frac{d}{dz} \begin{bmatrix}
E_x \\
E_y \\
H_x \\
H_y
\end{bmatrix} = -j \frac{\omega}{c} M \begin{bmatrix}
E_x \\
E_y \\
H_x \\
H_y
\end{bmatrix},
$$

(1)

where $M$ is a $4 \times 4$ matrix that depends on the permittivity and permeability tensors ($\varepsilon, \mu$) of the medium and of the tangential wave vector components at the frequency of the incident wave. $\omega$ is the angular frequency of the incident plane wave, and $c$ is the speed of light in vacuum.

The right/left circularly polarized (RCP/LCP) fields can be expressed in terms of the linear field components as follows:

$$
E^r = E_x + j E_y, \\
H^r = H_x + j H_y,
$$

(2)

and

$$
E^l = E_x - j E_y, \\
H^l = H_x - j H_y.
$$

(3)

For an imperfect circularly polarized incident wave, Eq. (2) is reformatted to consider the effect of deformed axial ratio as

$$
E^r = E_1 + j E_2, \\
H^r = H_1 + j H_2.
$$

(4)

where $E^r / H^r$ is the RCP wave and $E^l / H^l$ is the LCP wave.

The proposed method is based on using circular polarization measurements, but it is also correct for imperfect circular polarization.\footnote{29,30,31,32} For an imperfect circularly polarized incident wave, Eq. (2) is reformatted to consider the effect of deformed axial ratio as

$$
E^r = E_1 + j E_2, \\
H^r = H_1 + j H_2.
$$

(3)

where $E_1, E_2, H_1,$ and $H_2$ are given by

$$
E_1 = \alpha_1 e^{\varphi_1} E_0 e^{j \alpha_1} = \alpha_1 e^{\varphi_1} E_x, \\
E_2 = \alpha_2 e^{\varphi_2} E_0 e^{j \alpha_2} = \alpha_2 e^{\varphi_2} E_y, \\
H_1 = \alpha_1 e^\varphi_1 H_0 e^{j \alpha_1} = \alpha_1 e^\varphi_1 H_x, \\
H_2 = \alpha_1 e^\varphi_2 H_0 e^{j \alpha_2} = \alpha_1 e^\varphi_2 H_y,
$$

(4)

where $\alpha_1$, $\alpha_2$, $\beta_1,$ and $\beta_2$ are dispersive parameters that depend on the employed wave source.

For perfect circularly polarized sources, $\alpha_1 = \alpha_1$, and $\beta_1 = \beta_1$. Reordering Eq. (3) yields

$$
\begin{bmatrix}
E^r \\
E^l \\
H^r \\
H^l
\end{bmatrix} = \begin{bmatrix}
\alpha_1 e^{\varphi_1} & \alpha_1 e^{\varphi_2} & 0 & 0 \\
\alpha_1 e^{\varphi_2} & -\alpha_1 e^{\varphi_1} & 0 & 0 \\
0 & 0 & \alpha_2 e^{\varphi_1} & j \alpha_2 e^{\varphi_2} \\
0 & 0 & j \alpha_2 e^{\varphi_2} & -\alpha_2 e^{\varphi_1}
\end{bmatrix} \begin{bmatrix}
E_x \\
E_y \\
H_x \\
H_y
\end{bmatrix}.
$$

(5)

Let

$$
J = \begin{bmatrix}
\alpha_1 e^{\varphi_1} & j \alpha_2 e^{\varphi_2} & 0 & 0 \\
\alpha_1 e^{\varphi_2} & -\alpha_1 e^{\varphi_1} & 0 & 0 \\
0 & 0 & \alpha_2 e^{\varphi_1} & j \alpha_2 e^{\varphi_2} \\
0 & 0 & j \alpha_2 e^{\varphi_2} & -\alpha_2 e^{\varphi_1}
\end{bmatrix}.
$$

Then, Eq. (1) can be rewritten as

$$
\frac{d}{dz} \begin{bmatrix}
E^r \\
E^l \\
H^r \\
H^l
\end{bmatrix} = -j \frac{\omega}{c} JM^{-1} \begin{bmatrix}
E^r \\
E^l \\
H^r \\
H^l
\end{bmatrix}.
$$

(6)
For a general anisotropic medium, the elements of the $4 \times 4$ matrix $M$ are given as $^{11}$

$$M = \begin{bmatrix} M_1 & M_2 \end{bmatrix},$$

where $M_1$ and $M_2$ are $4 \times 2$ submatrices, given as

$$M_1 = \begin{bmatrix}
k_x \frac{\omega}{c} + k_y \frac{\mu}{\mu_0} & -k_y \left( k_y \frac{\mu}{\mu_0} + \frac{\omega}{c} \right) \\
-k_y \left( k_y \frac{\mu}{\mu_0} - \frac{\omega}{c} \right) & k_x \frac{\omega}{c} + k_y \frac{\mu}{\mu_0}
\end{bmatrix},$$

$$M_2 = \begin{bmatrix}
\frac{\omega}{c} \varepsilon_{xx} - \omega k_x \frac{\varepsilon}{\varepsilon_0} & -\omega \varepsilon_{xy} - \omega k_y \frac{\varepsilon}{\varepsilon_0} + \frac{\omega}{c} \varepsilon_{xx} \\
-k_y \frac{\omega}{c} \varepsilon_{yy} - k_x \frac{\varepsilon}{\varepsilon_0} & -k_x \frac{\omega}{c} \varepsilon_{yy} + k_y \frac{\varepsilon}{\varepsilon_0} \varepsilon_{xx}
\end{bmatrix},$$

where $k_x$ and $k_y$ are the wavevectors parallel to the anisotropic layer.

The solution of Eq. (6) is given as

$$\begin{bmatrix} E'(d) \\ E'(d) \\ H'(d) \\ H'(d) \end{bmatrix} = \exp \left( -j \frac{\omega}{c} MJ^{-1} d \right) \begin{bmatrix} E'(0) \\ E'(0) \\ H'(0) \\ H'(0) \end{bmatrix}. \quad (8)$$

The tangential field components at the interfaces of the biaxial anisotropic slab are continuous; thus, the fields inside the slab at distance $d$ can be found using the transfer matrix $T_r(0, d)$ as follows:

$$T_r(0, d) = \exp \left( -j \frac{\omega}{c} MJ^{-1} d \right). \quad (9)$$

Part of the normal-incident circularly polarized waves will be transmitted to the other side of the slab, and the other part will be reflected. The transmitted and the reflected fields can be determined from the transfer matrix as follows:

$$S_{11} \nu_r = T_r(0, d) (\nu_r + S_{11} \nu_r), \quad (10)$$

where $\nu_r$ and $\nu_r$ are the incident and reflected circularly polarized fields, respectively. $T_r(0, d)$ is the transfer matrix of the biaxial anisotropic slab given in Eq. (9). $S_{11}$ and $S_{22}$ are the circular reflection and circular transmission coefficients that can be determined from Eq. (10).

In general, a homogeneous slab of the biaxial anisotropic material oriented in the coordinate directions has constitutive electromagnetic parameters given by

$$\begin{bmatrix} D_1 \\ D_2 \\ D_3 \end{bmatrix} = \begin{bmatrix} \varepsilon_x & 0 & 0 \\ 0 & \varepsilon_y & 0 \\ 0 & 0 & \varepsilon_z \end{bmatrix} \begin{bmatrix} E_1 \\ E_2 \\ E_3 \end{bmatrix}, \quad (11)$$

$$\begin{bmatrix} B_1 \\ B_2 \\ B_3 \end{bmatrix} = \begin{bmatrix} \mu_x & 0 & 0 \\ 0 & \mu_y & 0 \\ 0 & 0 & \mu_z \end{bmatrix} \begin{bmatrix} H_1 \\ H_2 \\ H_3 \end{bmatrix}. \quad (12)$$

From Eq. (4), Eqs. (11) and (12) can be rewritten as

$$\begin{bmatrix} a_1 e^{\theta_1} B_x \\ a_2 e^{\theta_2} B_y \\ a_3 e^{\theta_3} B_z \end{bmatrix} = \begin{bmatrix} \varepsilon_x & 0 & 0 \\ 0 & \varepsilon_y & 0 \\ 0 & 0 & \varepsilon_z \end{bmatrix} \begin{bmatrix} a_1 e^{\theta_1} E_x \\ a_2 e^{\theta_2} E_y \\ a_3 e^{\theta_3} E_z \end{bmatrix}, \quad (13)$$

$$\begin{bmatrix} \mu_x & 0 & 0 \\ 0 & \mu_y & 0 \\ 0 & 0 & \mu_z \end{bmatrix} \begin{bmatrix} \mu_x & 0 & 0 \\ 0 & \mu_y & 0 \\ 0 & 0 & \mu_z \end{bmatrix} \begin{bmatrix} a_1 e^{\theta_1} H_x \\ a_2 e^{\theta_2} H_y \\ a_3 e^{\theta_3} H_z \end{bmatrix}. \quad (14)$$

The permittivity and permeability tensors for biaxial anisotropic materials are diagonal; this yields that Eqs. (13) and (14) can be rewritten as

$$\begin{bmatrix} D_1 \\ D_2 \\ D_3 \end{bmatrix} = \begin{bmatrix} \varepsilon_x & 0 & 0 \\ 0 & \varepsilon_y & 0 \\ 0 & 0 & \varepsilon_z \end{bmatrix} \begin{bmatrix} E_1 \\ E_2 \\ E_3 \end{bmatrix}, \quad (15)$$

$$\begin{bmatrix} B_1 \\ B_2 \\ B_3 \end{bmatrix} = \begin{bmatrix} \mu_x & 0 & 0 \\ 0 & \mu_y & 0 \\ 0 & 0 & \mu_z \end{bmatrix} \begin{bmatrix} H_1 \\ H_2 \\ H_3 \end{bmatrix}. \quad (16)$$

where $\varepsilon_x$, $\varepsilon_y$, and $\varepsilon_z$ are the relative permittivities and $\mu_x$, $\mu_y$, and $\mu_z$ are the relative permeabilities in the $xyz$ principal optical axes.

In this paper, the permittivity and permeability tensors of a biaxial anisotropic slab are extracted from the free space reflection and transmission data at $N$ distinct frequency points. In order to determine all tensor elements, three different slab orientations are required, as shown in Fig. 1. Theoretically, the orientations of the slab axes are described by Euler’s angles, $^3\theta, \phi, \psi$, with respect to the fixed $xyz$ coordinates.

![FIG. 1. Three orthogonal orientations of the biaxial anisotropic slab with respect to fixed $xyz$ coordinates: (a) the first position of the biaxial anisotropic slab, (b) the second position of the biaxial anisotropic slab, and (c) the third position of the biaxial anisotropic slab.](image-url)
The dielectric tensors after rotation in the \(xyz\) coordinate system are given by

\[
\bar{\varepsilon} = R \begin{bmatrix} \varepsilon_x & 0 & 0 \\ 0 & \varepsilon_y & 0 \\ 0 & 0 & \varepsilon_z \end{bmatrix} R^{-1},
\]

(17)

\[
\bar{\mu} = R \begin{bmatrix} \mu_x & 0 & 0 \\ 0 & \mu_y & 0 \\ 0 & 0 & \mu_z \end{bmatrix} R^{-1},
\]

(18)

where \(R\) is the coordinate rotation matrix given by

\[
R = \begin{bmatrix} 
\cos \psi \cos \theta - \cos \theta \sin \phi \sin \psi & -\sin \psi \cos \phi - \cos \theta \sin \phi \cos \psi & \sin \theta \sin \phi \\
\cos \psi \sin \phi + \cos \theta \cos \phi \sin \psi & -\sin \psi \sin \phi + \cos \theta \cos \phi \sin \psi & -\sin \theta \cos \phi \\
\sin \theta \sin \psi & \sin \theta \cos \psi & \cos \theta 
\end{bmatrix},
\]

(19)

Fresnel–Airy formulas relate the complex wave impedance and the complex refractive index to the reflection and transmission coefficients as

\[
z = \pm \sqrt{\frac{1 + S_{11}^2 - S_{21}^2}{1 - S_{11}^2 - S_{21}^2}},
\]

(20)

\[
n = -\frac{j}{k_o d} \ln \left( \frac{S_{11}}{1 - S_{21} (\frac{z - 1}{2z})} \right).
\]

(21)

For passive materials, the real value of \(z\) and the imaginary part of the refractive index, \(n\), must be greater than or equal to zero. Therefore, the sign of \(z\) is determined according to these conditions.

III. RESULTS AND DISCUSSIONS

A lossless homogeneous biaxial anisotropic slab aligned with the \(xyz\) axis with 8-\(\mu\)m thickness (in the \(z\) direction) is investigated, as shown in Fig. 1. The parameters of the tensors of the electrical permittivity and magnetic permeability for the lossless biaxial anisotropic slab are defined as functions of frequency, shown in Fig. 2, to ensure more general inspection of the proposed method.

Figure 3 shows comparisons between the extracted permittivity and permeability tensors of the lossless biaxial materials using the proposed method and those of recently published methods based on data obtained from linearly polarized waves at four measurement positions. The proposed technique uses data obtained from circularly polarized waves at three measurement positions only. The permittivity and permeability tensors for the lossless anisotropic slab are calculated as shown in Fig. 3. Apparent discontinuities in the permittivity and permeability curves ensued with linear polarization data that make the estimation of the proper parameters impractical at any considerable frequency range. This discontinuity is even greater for lossless thick slabs at higher frequencies. For lossy materials, the discontinuities occur at high frequencies, particularly when the thicker-material slabs are investigated.

The frequency envelopes corresponding to the correct extracted parameters with frequency are preserved using the circular polarization method. The circular polarization method extracts all tensor elements exactly as the original predefined permittivities and permeabilities shown in Fig. 2. The values of the correctly extracted material constitutive parameters could be positive or negative for homogeneous and inhomogeneous periodic structure synthesized anisotropic materials.
FIG. 2. The predefined (a) permittivity and (b) permeability tensors of a lossless biaxial anisotropic slab.

FIG. 3. The extracted permittivity and permeability tensors of the proposed method and linear polarization based method. (a), (c), and (e) are the permittivity tensor elements. (b), (d), and (f) are the permeability tensor elements.
To check the validity of the extracted tensor data using the proposed method, the percentage of the estimation error for left-hand circularly polarized data is shown in Fig. 4. The maximum error in the extracted parameters is 1.1%, which mostly contributed to computation approximations.

IV. CONCLUSIONS

This paper presents an effective retrieval method for extracting the permittivity and permeability tensors of a general biaxial anisotropic material. The proposed method relies on using measurements of a simple normal-incident circularly polarized wave at only three positions. The basic advantage of the proposed method is that it overcomes the limitations in usable frequency band and material losses. These limitations, observed in the previously published methods, are imposed by the multi-branching discontinuities in the complex refractive index for lossless anisotropic materials over the entire frequency band and for lossy materials at higher frequency bands, especially for thick slabs. The results obtained from the proposed method are accurate and trace the correct positive and negative permeability and permittivity tensor values over the entire frequency band for arbitrary slab thickness. This method is a direct-extraction mathematical procedure that avoids any heavy computations or optimization schemes. The retrieved tensor elements are validated by solving the original electromagnetic problem analytically by extending the Berreman $4 \times 4$ matrix method to the case of circularly polarized fields. The error in the extracted values is less than 1.1%, which is contributed by the round-offs in the mathematical computations.

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