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Susceptible exposed infected recovery (SEIR) model with immigration: equilibria points and its application

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Abstract. Infectious diseases such as rubella, mumps, measles, pertussis are caused by viruses or bacteria. There is a period of incubation in the spread of these diseases. The incubation period is a period in which individuals who are attacked by viruses or bacteria began shows clinical symptoms but have not been able to transmit the disease. The spread of disease by observing the period of incubation can be represented by the susceptible exposed infected recovery (SEIR) model. Immigration has an influence on the spread of disease. This is caused by immigrants who could carry the disease from their areas to other places. For this reason, the SEIR model with immigration to be considered. Here we will describe the SEIR model with immigration, determine an equilibrium point and state the equilibrium stability. The model is then applied to the measles disease. The SEIR model with immigration can be represented as system of the first ordinary differential equations. The model has two endemic equilibrium points. It means that there is still disease at the equilibrium point. The criteria of the equilibrium points are unstable.

INTRODUCTION

Piccolo and Billings [8] stated that all countries still face childhood diseases such as measles, rubella, and mumps. There is a period of incubation in the spread of these diseases. For these diseases, after individuals get the initial infection, the host remains in a latent stage for a period of time before becoming infectious. Tuberculosis and chronic hepatitis are examples of diseases that have longer incubation period.

Mathematical models have become important tools in analyzing the spread and control of infectious diseases. Many studies on epidemic models often includes persistence and extinction the disease. Most of the models in mathematical epidemiology are compartments. For diseases that have longer incubation period, in some common researches a population is divided into susceptible, exposed, infective, and recovered compartments; the model is then called SEIR model.

Many researchers have been conducted on SEIR models. The SEIR model with varying total population size was discussed by Li et al. [6], with limited resource for treatment was analyzed by Al-Sheikh [1], and with varying population size and vaccination was analyzed by Sun and HenHsieh [9].

Based on SEIR model Azizah et al. [2] discussed spread of ebola with isolation treatment and its analogy with rehabilitation treatment in drug abuse was discussed by Sutanto et al. [10].

Diseases may be introduced into a population by arrival of infected individuals from outside of the population. Travelers may return home from a foreign trip, for instant, with an infection acquired abroad. Influenza and measles are two examples of diseases that can easily spread between regions (countries or cities) due to travel. Juan et al. [5] and Hia et al. [4] stated that immigration influence diseases. Wang et al. [13] have investigated the disease transmission model by considering population immigration.

SEIR MODEL

A construction mathematical SEIR model is based on Tessa [11]. The population is divided into four compartments: susceptible, exposed, infectious, and recovered. The susceptible individuals are people who can be transmitted the disease, exposed individuals are people who their body is a host for infectious but are not yet

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able to transmit the disease, infectious individuals are people who have the disease and can transmit the disease, and recovered individuals are people who have recovered from the disease. The number of susceptible, exposed, infected, and recovered at time t respectively are denoted by *S(t)*, *E(t)*, *I(t)*, and *R(t)*. The size of population is assumed to be constant as much as N, so *S(t)* + *E(t)* + *I(t)* + *R(t)* = N.

Tessa [10] assumed that the birth rate is equal to the death rate, μ. If new individuals born are categorized as susceptible, the number of susceptible increases as many μN individuals. The transmission disease can only occur through a direct contact between susceptible and infected individual. If β is the effective contact rate, as many $\beta S \frac{I}{N}$ susceptible will be exposed. The instantaneous rate of change on the number of individuals infected can be expressed as

$$\frac{dS}{dt} = \mu N - \beta S \frac{I}{N} - \mu S. \tag{1}$$

The exposed individuals will be infected at the end of a latent period. If σ is the rate at which the exposed individuals become infective so that $1/\sigma$ the mean latent period, as many σE exposed will be infected. The number of exposed individuals increases as many $\beta S \frac{I}{N}$ individuals. The instantaneous rate of change on the number of individuals exposed can be expressed as

$$\frac{dE}{dt} = \beta S \frac{I}{N} - \mu E - \sigma E. \tag{2}$$

The number of infected individuals increases, as many σE, because of the existence exposed individuals whose have infected. Infected individuals who could hold out and have immunity will recover. If is the rate for recovery, the number of recovered individuals is γE. The number of recovered individuals, as many γI, increases because infected individuals who have recovered. The instantaneous rate of change on the number of individuals infected and recovered can be expressed as

$$\begin{aligned} \frac{dI}{dt} &= \sigma E - \gamma I - \mu I \\ \frac{dR}{dt} &= \gamma I - \mu R. \end{aligned} \tag{3}$$

Based on equations (1), (2), and (3), SEIR model can be rewritten as

$$\begin{aligned} \frac{dS}{dt} &= \mu N - \beta S \frac{I}{N} - \mu S \\ \frac{dE}{dt} &= \beta S \frac{I}{N} - \mu E - \sigma E \\ \frac{dI}{dt} &= \sigma E - \gamma I - \mu I \\ \frac{dR}{dt} &= \gamma I - \mu R. \end{aligned} \tag{4}$$

SEIR MODEL WITH IMMIGRATION

In this section we discuss a formulation of the immigration impact for outbreak the disease building on SEIR model. The formula refers to Zhang et al. [5] and Hia et al. [4]. However, here there is no individuals those dead because of the disease and the population is assumed to be constant. We assume that an individual may be infected only through contacts with infectious individuals.

In this model we assume that new individuals of immigration are all infected. If η is the immigrant rate, the number of infected individuals increase as much ηN. Because we assume that population is constant, the rate of

natural death at each compartment with formerly is assumed to be μ change to be $(\mu+\eta)$. The dynamical population SEIR model (4) with immigration is demonstrated in Figure 1.

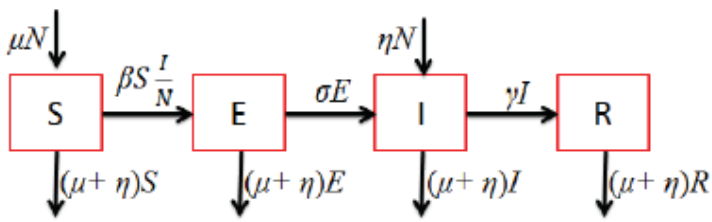

FIGURE 1. Dynamical population of SEIR model with immigration

Using Figure 1, SEIR model with immigration is represented as

$$\begin{aligned}
\frac{dS}{dt} &= \mu N - \beta S\frac{I}{N} - (\mu+\eta)S \\
\frac{dE}{dt} &= \beta S\frac{I}{N} - (\mu+\eta)E - \sigma E \\
\frac{dI}{dt} &= \sigma E + \eta N - \gamma I - (\mu+\eta)I \\
\frac{dR}{dt} &= \gamma I - (\mu+\eta)R.
\end{aligned} \tag{5}$$

where S(0)>0, E(0)≥0, I(0) >0, R(0) ≥ 0 and parameter μ, β, γ, η, σ, have positive values. These parameters are successively the rate of natural death, contact rate, rate for recovery, immigrant rate, and latent period. The solution of system (5) is the pattern of spread the diseases.

EQUILIBRIA POINTS

A system has an equilibrium point if there is no change in the system at all the time, Meyer [7]. Therefore, equilibria for system (5) can be found by setting the right sides of the four differential equations of (5) equal to zero. This is giving the nonlinear system

$$\begin{aligned}
\mu N - \beta S\frac{I}{N} - (\mu+\eta)S &= 0 \\
\beta S\frac{I}{N} - (\mu+\eta)E - \sigma E &= 0 \\
\sigma E + \eta N - \gamma I - (\mu+\eta)I &= 0 \\
\gamma I - (\mu+\eta)R. &= 0
\end{aligned} \tag{6}$$

The solution of system (6) is an equilibrium point. There are two equilibria points of system (6).

(i) Point $E_1 = (S_1^*, E_1^*, I_1^*, R_1^*)$ where

$$S_1^* = \frac{N}{2\beta\sigma}(\beta\eta^2 + \gamma\eta^2 + \eta^3 + \beta\eta\mu + 2\gamma\eta\mu + 3\eta^2\mu + \gamma\mu^2 + 3\eta\mu^2 + \mu^3 +$$
$$\beta\eta\sigma + \gamma\eta\sigma + \eta^2\sigma + \beta\mu\sigma + \gamma\mu\sigma + 2\eta\mu\sigma + \mu^2\sigma + A),$$
$$E_1^* = -\frac{N}{2\beta\sigma(\mu+\eta+\sigma)}(\beta\eta^2 + \gamma\eta^2 + \eta^3 + \beta\eta\mu + 2\gamma\eta\mu + 3\eta^2\mu + \gamma\mu^2 + 3\eta\mu^2 + \mu^3 + \beta\eta\sigma +$$
$$\gamma\eta\sigma + \eta^2\sigma - \beta\mu\sigma + \gamma\mu\sigma + 2\eta\mu\sigma + \mu^2\sigma + A),$$
$$I_1^* = -\frac{N}{2\beta(\mu+\eta+\sigma)(\mu+\eta+\gamma)}(-\beta\eta^2 + \gamma\eta^2 + \eta^3 - \beta\eta\mu + 2\gamma\eta\mu + 3\eta^2\mu + \gamma\mu^2 +$$
$$3\eta\mu^2 + \mu^3 - \beta\eta\sigma + \gamma\eta\sigma + \eta^2\sigma - \beta\mu\sigma + \gamma\mu\sigma + 2\eta\mu\sigma + \mu^2\sigma + A),$$
$$R_1^* = -\frac{N}{2\beta(\mu+\eta+\sigma)(\mu+\eta+\gamma)(\mu+\eta)}(-\beta\eta^2 + \gamma\eta^2 + \eta^3 - \beta\eta\mu + 2\gamma\eta\mu + 3\eta^2\mu +$$
$$\gamma\mu^2 + 3\eta\mu^2 + \mu^3 - \beta\eta\sigma + \gamma\eta\sigma + \eta^2\sigma - \beta\mu\sigma + \gamma\mu\sigma + 2\eta\mu\sigma + \mu^2\sigma + A),$$

and

$$A = \frac{\sqrt{N^2(\eta+\mu)(4\beta\eta(\gamma+\eta+\mu)(\eta+\mu+\sigma)^2 + (\eta+\mu)(\beta(\eta+\sigma) - (\gamma+\eta+\mu)(\eta+\mu+\sigma))^2)}}{N}.$$

The value of $I_1^* = 0$. It means that the spread of diseases still exist all the time or the system is on an the endemic state. So, E_1 is the endemic equilibrium point.

(ii) Point $E_2 = (S_2^*, E_2^*, I_2^*, R_2^*)$ where

$$S_1^* = \frac{N}{2\beta\sigma}(\beta\eta^2 + \gamma\eta^2 + \eta^3 + \beta\eta\mu + 2\gamma\eta\mu + 3\eta^2\mu + \gamma\mu^2 + 3\eta\mu^2 + \mu^3 +$$
$$\beta\eta\sigma + \gamma\eta\sigma + \eta^2\sigma + \beta\mu\sigma + \gamma\mu\sigma + 2\eta\mu\sigma + \mu^2\sigma - A),$$
$$E_1^* = -\frac{N}{2\beta\sigma(\mu+\eta+\sigma)}(\beta\eta^2 + \gamma\eta^2 + \eta^3 + \beta\eta\mu + 2\gamma\eta\mu + 3\eta^2\mu + \gamma\mu^2 + 3\eta\mu^2 + \mu^3 + \beta\eta\sigma +$$
$$\gamma\eta\sigma + \eta^2\sigma - \beta\mu\sigma + \gamma\mu\sigma + 2\eta\mu\sigma + \mu^2\sigma - A),$$
$$I_1^* = -\frac{N}{2\beta(\mu+\eta+\sigma)(\mu+\eta+\gamma)}(-\beta\eta^2 + \gamma\eta^2 + \eta^3 - \beta\eta\mu + 2\gamma\eta\mu + 3\eta^2\mu + \gamma\mu^2 +$$
$$3\eta\mu^2 + \mu^3 - \beta\eta\sigma + \gamma\eta\sigma + \eta^2\sigma - \beta\mu\sigma + \gamma\mu\sigma + 2\eta\mu\sigma + \mu^2\sigma - A),$$
$$R_1^* = -\frac{N}{2\beta(\mu+\eta+\sigma)(\mu+\eta+\gamma)(\mu+\eta)}(-\beta\eta^2 + \gamma\eta^2 + \eta^3 - \beta\eta\mu + 2\gamma\eta\mu + 3\eta^2\mu +$$
$$\gamma\mu^2 + 3\eta\mu^2 + \mu^3 - \beta\eta\sigma + \gamma\eta\sigma + \eta^2\sigma - \beta\mu\sigma + \gamma\mu\sigma + 2\eta\mu\sigma + \mu^2\sigma - A),$$

The value of I_2^* So the second point E_2 is also the endemic equilibrium point.

STABILITY OF EQUILIBRIA POINTS

Based on Bellomo dan Preziosi [3], to know the behavior of the system when there is a slight disturbance or change in the system and its influence on the equilibrium point is used the concept of stability point of equilibrium. System (5) is nonlinear, to observe stability of the equilibria points of the system is difficult. According to Bellomo and Preziosi [3], the system (5) should be linearized. The criteria of stability for the linear system with *n* equations is written in the following theorem.

Theorema 5.1. Suppose λ_i be eigen values of Jacobian matrix of the linear system which are evaluated at the equilibrium point x^* and $R_e(\lambda_i)$ is a real part of λ_i.

(1) If $\forall_i$ for $i = 1, \ldots, n$ $R_e(\lambda_i)$ has negative values, x^* is asymptotic stable, and

(2) if $\exists_i$ so that $R_e(\lambda_I)$ has positive values then x^* is unstable.

To apply Theorem 5.1 to the linearized system (5), we should determine that eigen values of Jacobian matrix evaluated at the equilibrium point is positive or negative. For this reason, Descartes's rule of signs could be used, Wang [14].

(1) Stability of the equilibrium point E_i could be determined from eigen values of characteristic equation

$$p_1(\lambda) = (-\lambda - \mu - \eta)(a\lambda^3 + b\lambda^2 + c\lambda + d) \tag{7}$$

From equation (7) could be shown that values of *a, b, c*< 0. Based on Descartes's rule of signs, $(a\lambda^3 + b\lambda^2 + c\lambda + d)$ has a positive eigen value. So using Theorem 5.1 the equilibrium point E_1 is unstable.

(2) Stability of the equilibrium point E_2 could be determined from eigen values of characteristic equation

$$p_2(\lambda) = (-\lambda - \mu - \eta)(-\lambda^3 + l\lambda^2 + m\lambda + n) \tag{8}$$

From equation (8) could be shown that values *l, m, n* < 0. Therefore, $(-\lambda^3 + l\lambda^2 + m\lambda + n)$ has three eigen values with the real part is negative. For $(-\lambda - \mu - \eta)$, we found a negative eigen value. Based on Theorem 5.1, the equilibrium point E_2 is asymptotic stable.

APPLICATION

Here the model (5) is applied to the spread of measles with the values of parameter are based on Troitter and Phillipe [12]. The size of population is 100000 people, the effective contact rate $\beta = 0.0000214$. The birth rate, μ, is 0.00875. The latent period and the infectious period of measles are 4-12 days and 17-31 days (Sun and HenHsien [9]), so the average of latent period and average of infectious period of the disease are 8 days and 24 days. The infectivity rate σ is 1/8 and the recovery rate is 1/24. It is assumed that the immigrant rate η is 0.000215 and the death rate $(\mu + \eta)$ is 0.008965. This application is the same as its by Widyaningsih and Nugroho [14] except the average of infectious period. For these parameters, model SEIR with immigration (5) for measles could be represented as

$$\begin{aligned}
\frac{dS}{dt} &= 0.00875N - 0.0000214S\frac{1}{100000} - 0.008965S \\
\frac{dE}{dt} &= 0.0000214S\frac{1}{100000} - 0.008965E - \frac{1}{8}E \\
\frac{dI}{dt} &= \frac{1}{8}E + 0.000215N - 0.008965I - \frac{1}{24}I \\
\frac{dR}{dt} &= \frac{1}{24}I - 0.008965R
\end{aligned} \tag{9}$$

The solution of system (9) was estimated using the four-order Runge-Kutta algorithm with the initial-values $S(0) = 68500$; $E(0) = 12000$; $I(0) = 19500$, and $R(0) = 0$. Figure 2 and 3 are the solution for $0 \le t \le 1300$ days.

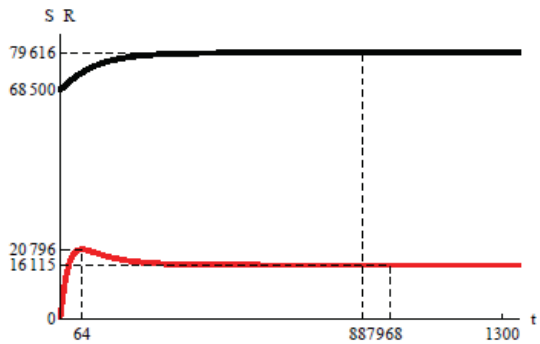

FIGURE 2. The number of susceptible(S), recovered (R) individuals for the first 1300 days.

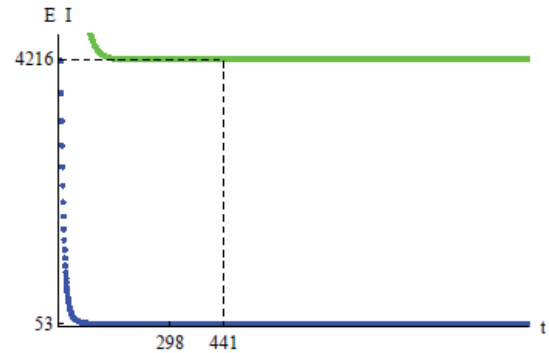

FIGURE 3. The number of exposed (E) and infected (I) individuals for the first 1300 days

From Figure 2 it appears that the number of susceptible individuals (S) increases from its initial value (68500 people) to the 887th day (79616 people). After the day, this number still remain the same. The number of recovered individuals also increases and achieves its peak at the 64th (20796 people). This number then decreases and is not vary in time start from the 968th (16115 people). From Figure 3, the number of exposed and infected individuals decreases from initial time to the 298th and 441th (12000 to 53 people and 195000 to 4216 people). After the time, this number unchange. By observing the patterns of measles spread in the four compartments, every compartment start to equilibrate at the 968th with its equilibrium point is $E_2 = (S_2^*, E_2^*, I_2^*, R_2^*) =$ (79616, 53, 4216, 16115). At this point, the number of infected individuals (4216 people) still exist, so this is the endemic equilibrium point. The eigen values of characteristic equation~(\ref{jacob2a}) evaluated at the endemic equilibrium point E_2 are $\lambda_1 = -0.133991$, $\lambda_2 = -0.0506061$, $\lambda_3 = -0.0089659$, and $\lambda_4 = -0.008965$. Based on Theorem 5.1 the endemic equilibrium point E_2 = (79616, 53, 4216, 16115) is asymptotic stable.

CONCLUSION

(i) The model SEIR with immigration (5) has two endemic equilibria points.

(ii) From the application on the measles disease with μ =0.00875, β =0.00214, η =0.00215, σ =1/8, γ =1/24, and the initial values $S(0)$ = 68500, $E(0)$ = 12000, $I(0)$ = 19500, and $R(0)$ = 0, the spread of measles has the endemic equilibrium population with the number of infected individuals is 4216 people. This equilibrium is asymptotic stable.

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