Soliton-generating $\tau$-functions revisited

Cite as: J. Math. Phys. 59, 122701 (2018); https://doi.org/10.1063/1.5046356
Submitted: 27 June 2018. Accepted: 05 November 2018. Published Online: 03 December 2018

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In the inverse-scattering formalism and the Hirota algorithm, soliton solutions of evolution equations are images of \( \tau \)-functions, which are, often, unbounded in space and time. Hence, whereas solitons are bounded, localized structures, they are generated from unbounded, spatially extended entities that lack physically intuitive significance. The goal of this paper is to provide a representation of soliton solutions as images of bounded and localized sources. To this end, appropriate equivalent \( \tau \)-functions, which are different in form from the traditionally used ones, are generated. In terms of the latter, the single-soliton solutions of the Korteweg-deVries (KdV) and Kadomtsev-Petviashvili II (KP II) equations are images of single solitons (NOT solutions of theses equations), whereas multi-soliton solutions are images of positive humps that are localized in the soliton interaction region. The structure of the equivalent \( \tau \)-functions has another desirable attribute: It elucidates in a simple manner the role of the arbitrary shifts in the positions of soliton trajectories in controlling the behavior of multi-soliton solutions: Whether an \( N \)-soliton solution is reduced to a solution with \( (N-1) \) or \( (N-2) \) solitons when two wave numbers are made to coincide. Examples are discussed in the cases of the KdV and KP II equations. The modified KdV equation provides a unique observation. Depending on the shifts in the positions of soliton trajectories, when two wave numbers are made to coincide, the two-soliton solution may tend either to a single soliton or to a \( \delta \)-function in the \( x-t \) plane. © 2018 Author(s). All article content, except where otherwise noted, is licensed under a Creative Commons Attribution (CC BY) license (http://creativecommons.org/licenses/by/4.0/). https://doi.org/10.1063/1.5046356

I. INTRODUCTION

Many well-known nonlinear evolution equations provide approximate descriptions of phenomena in physical systems. For example, the Korteweg-deVries (KdV) equation describes the propagation of waves in \( (1+1) \) dimensions on the surface of a shallow water layer,\(^1,2\) along a Fermi-Pasta-Ulam chain,\(^3\) and of ion acoustic waves in plasma physics;\(^4,5\) the Kadomtsev-Petviashvili II (KP II) equation describes the propagation of waves in \( (1+2) \) dimensions on the surface of a shallow water layer.\(^6\)

In the inverse-scattering\(^7-15\) and Hirota approaches,\(^16-21\) soliton solutions of integrable evolution equations are transforms of \( \tau \)-functions. Typically, the latter are expressed in terms of exponentials, the arguments of which are linear in the coordinates. Often, the \( \tau \)-functions are extended and unbounded in space and time, whereas their images, the soliton solutions, are bounded and localized structures.

The goal of this paper is to obtain a representation of soliton solutions as images of bounded and spatially localized sources. This is attained though exploitation of the non-uniqueness of the \( \tau \)-functions in order to construct \( \tau \)-functions, which are equivalent to the ones traditionally used in the inverse-scattering/Hirota formulation. Namely, they generate the same soliton solutions. Using
the equivalent $\tau$-functions, it is shown that the single soliton solutions of the KdV and KP II equations are images of single solitons (NOT a solution of these equations) and multi-soliton solutions are the image of positive humps that are localized in the region of soliton interaction.

The non-uniqueness of $\tau$-functions has been exploited in, probably, tens if not hundreds of papers. In the case of the KdV equation, the resulting equivalent $\tau$-function arrived at in this paper was first presented and exploited in the study of the tropical limit of the KdV equation.\textsuperscript{22}

The structure of the equivalent $\tau$-functions has another desirable attribute: It elucidates in a simple manner the role of the arbitrary shifts in the positions of soliton trajectories in space and time in controlling the limit of multi-soliton solutions when two wave numbers are made to coincide. Specifically, whether an $N$-soliton solution is reduced to a solution with $(N-1)$ or $(N-2)$ solitons. In the traditional forms of $\tau$-functions, this is not as transparent. Examples are discussed in the cases of the KdV and KP II equations.

The case of the modified KdV equation is particularly interesting. It is well known that the $\tau$-function arrived at in this paper was first presented and exploited in the study of the tropical limit of the KdV equation.

The general case of the KdV equation is discussed in the Appendix and Secs. II A and II B. Examples are discussed in Secs. II C–II F. The case of the KP II equation is discussed in Sec. III. The case of the modified KdV equation is discussed in Sec. IV.

Throughout the paper, any quantity associated with the usual formulation within the framework of the inverse-scattering/Hirota formalism will be called a “traditional” quantity.

II. THE KdV EQUATION

A. Motivation

The soliton solutions of the KdV equation,

$$u_t = 6uu_x + u_{xxx},$$

are constructed in terms of a $\tau$-function through\textsuperscript{10–17}

$$u(t, x) = 2\partial_x^{-2} \log \tau(t, x).$$

The traditional form of $\tau(t, x)$ for an $N$-soliton solution is\textsuperscript{10–17}

$$\tau_T(t, x) = 1 + \sum_{n=1}^{N} \sum_{1 \leq i_1 < i_2 < \cdots < i_n \leq N} \left( \prod_{j=1}^{n} \prod_{l=i+j+1}^{N} \left( \frac{k_{i_j} - k_l}{k_{i_j} + k_l} \right)^2 \right) e^{2 \sum_{m=1}^{n} \theta_{i_m}},$$

$$0 < k_1 < k_2 < \cdots < k_N$$

$$\theta_{i_l} = k_{i_l} x + \omega_{i_l} t + \delta_{i_l},$$

$$\omega_{i_l} = 4k_{i_l}^3.$$  \hspace{1cm} (3)

Here and in the following, the subscript $T$ denotes the traditional (inverse-scattering/Hirota) form.

In view of Eqs. (2) and (3), it is clear that the soliton solutions, which are bounded and spatially localized, are images of unbounded and spatially extended structures.

In addition, the role of the arbitrary shifts, $\delta_{i_l}$, in determining the properties of the solution, is not transparent. For specificity, let us discuss the two-soliton solution, for which Eq. (3) becomes

$$\tau_T(t, x) = 1 + e^{2\theta_1} + e^{2\theta_2} + \left( \frac{k_2 - k_1}{k_1 + k_2} \right)^2 e^{2\theta_1} e^{2\theta_2} \quad (k_2 > k_1 > 0).$$

Consider the case of constant shifts, $\delta_1$ and $\delta_2$. Trivially, in the limit when the wave numbers, $k_1$ and $k_2$, coincide, the two-soliton solution tends to a single-soliton one. However, this is not the only possibility. As $\delta_1$ and $\delta_2$ are arbitrary, assign them the following wave number dependence:
\[ \delta_1 = \gamma \ln \left( \frac{k_2 - k_1}{k_1 + k_2} \right), \quad \delta_2 = (1 - \gamma) \ln \left( \frac{k_2 - k_1}{k_1 + k_2} \right). \]  
(7)

Equation (6) then becomes
\[ \tau_T(t, x) = 1 + \left( \frac{k_2 - k_1}{k_1 + k_2} \right)^\gamma e^{2(k_1 x + \omega_1 t)} + \left( \frac{k_2 - k_1}{k_1 + k_2} \right)^{1 - \gamma} e^{2(k_2 x + \omega_2 t)} + \left( \frac{k_2 - k_1}{k_1 + k_2} \right) e^{2\theta_2 e^{2(k_2 x + \omega_2 t)}}. \]  
(8)

For any value of \( \gamma \), the resulting solution tends to zero when \( k_2 \to k_1 \).

While reaching this conclusion in the two-soliton case is relatively simple, deciphering the roles of the arbitrary shifts in the behavior of solutions with \( N > 2 \) solitons becomes far less transparent.

In the remainder of this paper, it will be shown that both issues discussed above can be taken care of through the non-uniqueness of \( \tau_T(t, x) \): Multiplying \( \tau_T(t, x) \) by any term of the form \( e^{g(t)x + f(t)} \) yields an equivalent \( \tau \)-function that generates the same solutions. An equivalent \( \tau \)-function will be presented, in terms of which one defines
\[ S(t, x) = \frac{1}{\tau_E(t, x)}. \]  
(9)

In Eq. (9), the subscript \( E \) stands for “equivalent.” \( S(t, x) \) will be shown to be bounded and localized in the \( x-t \) plane so that the solution of Eq. (1), expressed as
\[ u(t, x) = -2\partial_x^2 \log S(t, x), \]  
(10)

is now an image of a localized source.

The same choice of equivalent \( \tau \)-function also elucidates the role of the free shifts in soliton solutions in the changes in multi-soliton solutions when wave numbers are made to coincide.

**B. \( N \)-soliton solution**

**1. Search for a localized source**

In order to express \( u(t, x) \) in terms of a bounded and localized source as in Eqs. (9) and (10), one needs to find an appropriate equivalent \( \tau \)-function. To this end, consider an equivalent \( \tau \)-function
\[ \tau_E(t, x) = e^{-\sum_{i=1}^{N} \mu_i \theta_i} \tau(t, x). \]  
(11)

\( \tau_E(t, x) \) is a sum of \( 2^N \) terms, each a product of \( N \) exponentials. For \( S(t, x) \) of Eq. (9) to be bounded and localized in the \( x-t \) plane, one must ensure that \( \tau_E(t, x) \) is unbounded asymptotically in all directions in the plane. Namely, in any direction in the plane, some exponential terms in \( \tau_E(t, x) \) must become unbounded asymptotically. This can be achieved by choosing the coefficients, \( \mu_i \), so that the sum of the exponents of all the \( 2^N \) terms in Eq. (11) vanishes. Then, if in a domain in the plane some exponential terms in \( \tau_E(t, x) \) vanish asymptotically, others become exponentially unbounded.

The sum of all the exponents is
\[ 2^N \sum_{i=1}^{N} (1 - \mu_i) \theta_i. \]  
(12)

For this sum to vanish, the independence of \( \theta_i \) requires
\[ \mu_i = 1. \]  
(13)

The resulting equivalent \( \tau \)-function is (see the Appendix for the details of the transformation from \( \tau_T \) to \( \tau_E \))
where \( \delta_i \), the constant shifts in the traditional expression of Eq. (4), are replaced by wave number dependent shifts, \( \delta_i \). The traditional values of \( \Delta_i \) are determined by the transformation from \( \tau_T \) to \( \tau_E \). Examples are presented in Secs. II D–II F.

2. Peculiar properties of N-soliton solutions

An N-soliton solution, constructed from \( \tau_T \) of Eq. (3), is reduced to an \((N-1)\)-soliton solution when two wave numbers are made to coincide, provided all shifts, \( \delta_i \), are constant.

If \( \delta_i \) are wave-number dependent, the limit of the solution may be different. This characteristic is buried in the traditional form, Eq. (3), but is not transparent. Using the equivalent \( \tau \)-function, \( \tau_E \) of Eq. (14), the role played by \( \Delta_i(\vec{k}) \), the shifts in soliton trajectories, is elucidated directly.

The singular wave-number dependence of the traditional \( \Delta_i(\vec{k}) \) (obtained in the transformation from \( \tau_T \) to \( \tau_E \); see the Appendix and Secs. II C–II F) leads to the reduction of an N-soliton solution to an \((N-1)\)-soliton one. This can be seen directly from inspecting Eq. (14) with the aid of the Appendix. Examples of the two-, three-, and four-solitons are provided in Secs. II D–II F.

If \( \Delta_i(\vec{k}) \) lack this singular wave number dependence, then when \( k_2 \to k_1 \), the number of terms in \( \tau_E \) of Eq. (14) is reduced from \( 2^{N-1} \) to \( 2^{N-3} \), corresponding to an \((N-2)\)-soliton solution!

In Secs. II D–II F, it is shown that, when the \( \Delta_i(\vec{k}) \) are independent of the wave numbers, the limits of the two-, three-, and four-soliton solutions are, respectively, zero, one-, and two-soliton solutions.

With \( N \geq 4 \) solitons, there are more possibilities. If some \( \Delta_i(\vec{k}) \) are constant and others have the singular structure of the traditional case, then when some wave number pairs coincide, the solution is reduced to \((N-2)\) solitons, while for other pairs, it is reduced to \((N-1)\) solitons.

C. Single-soliton solution

This trivial case is discussed only so as to show the emerging pattern. The traditional \( \tau \)-function,

\[
\tau_T(t, x) = 1 + e^{2^2(ks+4k^3t+\delta)},
\]

where \( \delta \) is a constant shift, generates the single-soliton solution

\[
u(t, x) = \frac{2k^2}{(\cosh(kx + 4kt + \delta))^2}.\]

Multiplying the expression in Eq. (15) by \( e^{-(ks+4k^3t+\delta)/2} \) yields an equivalent \( \tau \)-function, which generates the same single-soliton solution through Eq. (2),

\[
\tau_E(t, x) = \cosh(kx + 4kt + \delta).
\]

Now define the source function of Eq. (9)

\[
S(t, x) = \frac{1}{\tau_E(t, x)} = \frac{1}{\cosh(kx + 4kt + \delta)}.
\]
The single soliton solution, generated by Eq. (10), is the image of the spatially bounded entity of Eq. (18), which is a single soliton (NOT a solution of the KdV equation).

D. Two-soliton solution

1. Equivalent \( \tau \)-function

Following the discussion in Sec. II A, the \( \tau \)-function that is equivalent to \( \tau_T \) of Eq. (6) is

\[
\tau_E = \frac{1}{2} \left( e^{-\theta_1} e^{-\theta_2} \right) \tau_T.
\]  

(19)

\( \tau_E \) is re-written as

\[
\tau_E = \left\{ \frac{(k_2 - k_1)(k_2 + k_1)}{(k_2 - k_1)(k_1 + k_2)} \cosh(\theta_{++} + \alpha) + \cosh \theta_{+-} \right\}.
\]  

(20)

In Eq. (20),

\[
\theta_{++} = \theta_1 + \theta_2, \quad \theta_{+-} = \theta_1 - \theta_2
\]  

(21)

and

\[
\sinh \alpha = -2k_1 k_2 / ((k_2 - k_1)(k_1 + k_2)).
\]  

(22)

2. Soliton trajectory shifts as solution parameters

In \( \tau_E \) of Eq. (20), the effect of \( \alpha \) can be translated into individual shifts of soliton trajectories,

\[
\tau_E(t, x) = \frac{(k_2 - k_1)(k_2 + k_1)}{(k_2 - k_1)(k_1 + k_2)} \cosh \tilde{\theta}_{++} + \cosh \tilde{\theta}_{+-},
\]

\[
\tilde{\theta}_{++} = \tilde{\theta}_1 + \tilde{\theta}_2, \quad \tilde{\theta}_{+-} = \tilde{\theta}_1 - \tilde{\theta}_2,
\]  

(23)

\[
\tilde{\theta}_i = \theta_i + (\alpha/2) = k_i x + \omega_i t + \Delta_i(\vec{k}), \quad \left( \Delta_i(\vec{k}) = \delta_i + (\alpha/2) \right).
\]

3. Localized source

Following Eq. (9), define a source

\[
S(t, x) = \frac{1}{\tau_E(t, x)} = \frac{1}{(k_2 - k_1)(k_2 + k_1) \cosh \tilde{\theta}_{++} + \cosh \tilde{\theta}_{+-}}.
\]  

(24)

\( S(t, x) \) is bounded and localized in the \( x-t \) plane. Hence, under Eq. (10), the two-soliton solution is the image of a localized source. The peak of the source is at \( \tilde{\theta}_{++} = \tilde{\theta}_{+-} = 0 \), and its widths in the \( \tilde{\theta}_{++} \) and \( \tilde{\theta}_{+-} \)-directions are \( 2/(1 - (k_1/k_2)^2)^{1/2} \) and \( 2/(1 + (k_1/k_2)^2)^{1/2} \), respectively. Figure 1 shows a two-soliton solution, and Fig. 2 shows its source, \( S(t, x) \).

FIG. 1. Two-KdV-soliton solution \( \tau_E \) of Eq. (20). \( k_1 = 0.2; k_2 = 0.3; \delta_1 = \delta_2 = \alpha = 0. \)
4. Limit of $k_2 \rightarrow k_1$

The structure of the traditional $\tau_T$ of Eq. (17) forces the solution to be reduced to a single-soliton in the limit when $\delta_i$ are constants. Constructing the solution through $\tau_E$ of Eqs. (23), the limit depends on the value of $\alpha$. If $\alpha$ has the traditional singular wave number dependence of Eq. (22), then the limit of one soliton is attained. If $\alpha$ does not have this singular nature, the limit vanishes,

$$u(t,x)\big|_{k_2\rightarrow k_1} = \begin{cases} \frac{2k_1^2}{\cosh \left( k_1 x + 4k_1^3 t + \mu \right)} & \sinh \alpha = -\frac{2k_1k_2}{(k_2-k_1)(k_1+k_2)} \quad \alpha = \text{Const} \\ 0 & \end{cases} \quad (25)$$

E. Three-soliton solution

The traditional form of the three-soliton $\tau$-function (wave numbers $k_3 > k_2 > k_1 > 0$) is

$$\tau_T(t,x) = 1 + e^{2\theta_1} + e^{2\theta_2} + e^{2\theta_3} + \left( \frac{k_2-k_1}{k_1+k_2} \right)^2 e^{2\theta_1} + \left( \frac{k_3-k_1}{k_1+k_3} \right)^2 e^{2\theta_2} + \left( \frac{k_3-k_2}{k_2+k_3} \right)^2 e^{2\theta_3}.$$ \quad (26)

1. Equivalent $\tau$-function

Following the discussion in Sec. II B 1, replace Eq. (26) by

$$\tau_E(t,x) = \frac{1}{2} e^{-\theta_1} e^{-\theta_2} e^{-\theta_3} \tau_T(t,x). \quad (27)$$

Rearranging the terms yields

$$\tau_E(t,x) = \frac{(k_2-k_1)}{(k_2+k_1)} \frac{(k_3-k_1)}{(k_3+k_1)} \frac{(k_3-k_2)}{(k_3+k_2)} \cosh(\theta_{+++} + \alpha_{+++}) + \frac{(k_2-k_1)}{(k_2+k_1)} \cosh(\theta_{+++} + \alpha_{+++}) + \frac{(k_3-k_1)}{(k_3+k_1)} \cosh(\theta_{+++} + \alpha_{+++}) + \frac{(k_3-k_2)}{(k_3+k_2)} \cosh(\theta_{+++} + \alpha_{+++}). \quad (28)$$

In Eq. (28),

$$\theta_{\sigma_1\sigma_2\sigma_3} = \sum_{i=1}^{3} \sigma_i \theta_i, \quad (\sigma_i = \pm 1), \quad (29)$$
with \( \theta_i \) defined by Eq. (4).

The transformation from \( \tau_F \) of Eq. (26) to \( \tau_E \) dictates the values of the constant shifts, \( \alpha_{\sigma_1 \sigma_2 \sigma_3} \), of Eq. (28) to be

\[
\sinh \alpha_{+++} = -2 \frac{(k_1^2 k_2 + k_1 k_3^2 + k_2 k_3^2 + k_1 k_2 k_3)}{(k_2^2 - k_1^2)(k_3^2 - k_1^2)} \sinh \alpha_{++-} = -2 \frac{2k_1 k_2}{(k_2^2 - k_1^2)},
\]

\[
\sinh \alpha_{+-+} = -2 \frac{k_1 k_3}{(k_3^2 - k_1^2)}, \quad \sinh \alpha_{-++} = -2 \frac{2k_2}{(k_3^2 - k_2^2)}. \tag{30}
\]

2. Soliton trajectory shifts as solution parameters

It does not take much to show that \( \alpha_{\sigma_1 \sigma_2 \sigma_3} \) obey the constraint

\[
\alpha_{+++} + \alpha_{++-} = \alpha_{+-+} + \alpha_{-++}. \tag{31}
\]

Thus, only three of \( \alpha_{\sigma_1 \sigma_2 \sigma_3} \) are linearly independent. This constraint is identical in shape to the constraint obeyed by the four \( \theta_{\sigma_1 \sigma_2 \sigma_3} \),

\[
\theta_{+++} + \theta_{++-} = \theta_{+-+} + \theta_{-++}. \tag{32}
\]

Equation (32) is a trivial consequence of Eq. (29). This suggests a similar decomposition for \( \alpha_{\sigma_1 \sigma_2 \sigma_3} \),

\[
\alpha_{\sigma_1 \sigma_2 \sigma_3}(\vec{k}) = \sum_{i=1}^{3} \sigma_i \alpha_i(\vec{k}). \tag{33}
\]

Equation (28) can be re-written in a form that exhibits the role of the shifts in soliton trajectories in solution properties,

\[
\tau_E(t, x) = \frac{(k_2 - k_1)(k_3 - k_1)(k_3 - k_2)}{(k_2 + k_1)(k_3 + k_1)(k_3 + k_2)} \cosh \tilde{\theta}_{+++} + \frac{(k_2 - k_1)}{(k_2 + k_1)} \cosh \tilde{\theta}_{++-} + \frac{(k_3 - k_1)}{(k_3 + k_1)} \cosh \tilde{\theta}_{+-+} + \frac{(k_3 - k_2)}{(k_3 + k_2)} \cosh \tilde{\theta}_{-++}. \tag{34}
\]

with

\[
\tilde{\theta}_{\sigma_1 \sigma_2 \sigma_3} = \sum_{i=1}^{3} \sigma_i \tilde{\theta}_i, \quad (\sigma_i = \pm 1), \tag{35}
\]

\[
\tilde{\theta}_i = \theta_i + \epsilon_i = k_i x + \omega_i t + \Delta_i(\vec{k}), \quad (\Delta_i(\vec{k}) = \delta_i + \alpha_i(\vec{k})). \tag{36}
\]

3. Localized source

With \( \tau_E \) of Eq. (34), the source, \( S(t, x) \), defined by Eq. (9), is bounded and localized in the \( x-t \) plane. Hence, the three-soliton solution becomes the image of a localized source. Figure 3 shows a three-soliton solution, and Fig. 4 shows its source, \( S(t, x) \).

4. Limit of \( k_2 \to k_1 \)

When \( k_2 \to k_1 \), Eq. (26) becomes a traditional two-soliton \( \tau \)-function if the \( \delta_i \) are constants; the three-soliton solution is reduced to a two-soliton one. With \( \tau_E \) of Eq. (28) or (34), the role of the shifts of soliton trajectories in the \( x-t \) plane becomes clear. Using \( \tau_E \) of Eq. (28), the limit depends on \( \alpha_{\sigma_1 \sigma_2 \sigma_3} \). In the traditional case, the two-soliton limit is obtained owing to the singular nature of \( \alpha_{\sigma_1 \sigma_2 \sigma_3} \) of Eq. (30). However, when \( \alpha_{\sigma_1 \sigma_2 \sigma_3} \) do not obey Eq. (30), the limit is different,
FIG. 3. Three-KdV-soliton solution $[\tau_E$ of Eq. (27)], $k_1 = 0.25; k_2 = 0.35; k_3 = 0.45; \delta_1 = \delta_2 = \delta_3 = 0; \alpha_{++} = \alpha_{+-} = \alpha_{-+} = \alpha_{--} = 0$.

\[
\tau_E(t, x)|_{k_2 \to k_1} = (4((k_3 - k_1)/(k_3 + k_1)) \cosh \mu) \cosh(\theta_3 + \nu),
\]
\[
\mu = \delta_1 - \delta_2 + (\alpha_{++} + \alpha_{-+})/2, \quad \nu = \delta_1 - \delta_2 + (\alpha_{+} - \alpha_{--})/2.
\]

Equation (37) is a $\tau$-function that generates a single KdV-soliton solution, with wave number $k_3$.

F. Four-soliton solution

The traditional $\tau$-function for four-soliton solution (wave numbers $k_4 > k_3 > k_2 > k_1 > 0$) is

\[
\tau_F(t, x) = 1 + \sum_{i=1}^{4} e^{2\theta_i} + \sum_{i=1}^{4} \sum_{j=i+1}^{4} \left( \frac{k_i - k_j}{k_i + k_j} \right)^2 e^{2\theta_i} e^{2\theta_j} + \sum_{i=1}^{4} \sum_{j=i+1}^{4} \sum_{m=j+1}^{4} \left( \frac{k_i - k_j}{k_i + k_j} \right)^2 e^{2\theta_i} e^{2\theta_j} \left( \frac{k_m - k_i}{k_j + k_m} \right)^2 e^{2\theta_j} e^{2\theta_m} + \left( \frac{k_2 - k_1}{k_1 + k_2} \right)^2 \left( \frac{k_3 - k_1}{k_1 + k_3} \right)^2 \left( \frac{k_4 - k_1}{k_1 + k_4} \right)^2 \left( \frac{k_3 - k_2}{k_2 + k_3} \right)^2 \left( \frac{k_4 - k_2}{k_2 + k_4} \right)^2 \left( \frac{k_4 - k_3}{k_3 + k_4} \right)^2 \left( \frac{k_4 - k_4}{k_4 + k_4} \right)^2 e^{2\theta_i} e^{2\theta_j} e^{2\theta_m} e^{2\theta_4}.
\]

1. Equivalent $\tau$-function

Following the procedure delineated in the Appendix, one obtains an equivalent $\tau$-function

FIG. 4. $S(t, x)$, [Eq. (9)], the source of the three-KdV-soliton solution. The parameters are the same as in Fig. 3.
\[
\tau_E(t,x) = \frac{(k_2 - k_1)(k_3 - k_1)(k_4 - k_1)(k_3 - k_2)(k_4 - k_2)(k_4 - k_3)}{(k_2 + k_1)(k_3 + k_1)(k_4 + k_1)(k_3 + k_2)(k_4 + k_2)(k_4 + k_3)} \cosh(\theta_{+++} + \alpha_{+++}) + \\
\frac{(k_2 - k_1)(k_3 - k_1)(k_3 - k_2)}{(k_2 + k_1)(k_3 + k_1)(k_3 + k_2)} \cosh(\theta_{+++} + \alpha_{+++}) + \\
\frac{(k_2 - k_1)(k_4 - k_1)(k_4 - k_2)}{(k_2 + k_1)(k_4 + k_1)(k_4 + k_2)} \cosh(\theta_{+++} + \alpha_{+++}) + \\
\frac{(k_2 - k_1)(k_4 - k_1)}{(k_2 + k_1)(k_4 + k_1)} \cosh(\theta_{+++} + \alpha_{+++}) + \\
\frac{(k_3 + k_1)(k_4 + k_1)(k_4 + k_3)}{(k_3 + k_1)(k_4 + k_1)(k_4 - k_3)} \cosh(\theta_{+++} + \alpha_{+++}) + \\
\frac{(k_2 + k_1)(k_4 + k_3)}{(k_2 + k_1)(k_4 + k_3)} \cosh(\theta_{+++} + \alpha_{+++}) + \\
\frac{(k_3 - k_1)(k_4 - k_2)}{(k_3 - k_1)(k_4 - k_2)} \cosh(\theta_{+++} + \alpha_{+++}) + \\
\frac{(k_3 - k_2)(k_4 - k_2)(k_4 - k_3)}{(k_3 - k_2)(k_4 - k_2)(k_4 + k_3)} \cosh(\theta_{+++} + \alpha_{+++}) + \\
\frac{(k_3 + k_2)(k_4 + k_2)(k_4 + k_3)}{(k_3 + k_2)(k_4 + k_2)(k_4 + k_3)} \cosh(\theta_{+++} + \alpha_{+++}),
\]

\(\alpha_{i,j} = \frac{\theta}{4 \sigma}(\sigma_i \sigma_j), \quad \sigma_1 = +1, \sigma_{i>1} = \pm 1\).  

2. Soliton trajectory shifts as solution parameters

Following the procedure delineated in the Appendix, the structure of \(\tau_f\) of Eq. (38) dictates the expressions for \(\alpha_{i,j}\) in the traditional case. Furthermore, \(\alpha_{i,j}\) obey the following constraints:

\[
\begin{align*}
\alpha_{+++} &= \alpha_{+++} - \alpha_{++++} + \alpha_{+--+}, \\
\alpha_{++-} &= \alpha_{++-} = \alpha_{++++} - \alpha_{++++} + \alpha_{+--+}, \\
\alpha_{+-+} &= \alpha_{+-+} + \alpha_{+-+} - \alpha_{++++}, \\
\alpha_{++-} &= \alpha_{++-} = \alpha_{++++} - \alpha_{++++} + 2\alpha_{++++} - \alpha_{++++}.
\end{align*}
\]

As in the case of the three-soliton solution, the constraints of Eq. (41) are identical in shape to four constraints obeyed by \(\theta_{\varphi}\). The latter are a trivial consequence of Eq. (40). This allows, again, for the construction of the eight linearly dependent shifts in terms of four independent shifts

\[
\alpha_{i,j}(\vec{k}) = \sum_{i=1}^{4} \sigma_i \alpha_i(\vec{k})
\]

and for re-writing of Eq. (39) in the form of Eq. (14). Again, the parameters, on which the four-soliton solution depends, have been formulated as shifts of soliton trajectories in the \(x-t\) plane.

3. Localized source

Using Eq. (39) in the definition, Eq. (9), of \(S(t,x)\), the four-soliton solution becomes the image of a source, a hump that is localized in the \(x-t\) plane.

4. Limit of coinciding wave numbers

The structure of \(\tau_f\) of Eq. (38) ensures that when \(k_2 \to k_1\), if the \(\delta_i\) are constants, the solution is reduced to a three-soliton solution (wave numbers \(k_1, k_3, \) and \(k_4\)). If one next considers the limit of \(k_4 \to k_3\), then the three-soliton solution is reduced to a two-soliton solution (wave numbers \(k_1\) and \(k_3\)). If the solution is constructed from \(\tau_f\) of Eq. (39), the limit depends on \(\alpha_{i,j}\). If the latter assume the singular wave-number dependent values dictated by the structure of \(\tau_f\), then the above-mentioned limits are reached. If, however, \(\alpha_{i,j}\) do not have the singular behavior, the limit may be different.
If all $\alpha_\delta$ are constants, the limit, $k_2 \rightarrow k_1$, of the solution is a two-soliton solution (wave numbers $k_3$ and $k_4$). Imposing, in addition, $k_4 \rightarrow k_3$, this two-soliton solution is reduced to zero.

**III. THE KADOMTSEV-PETVIASHVILI II EQUATION**

The line-soliton solutions of the Kadomtsev-Petviashvili II (KP II) equation,

$$\frac{\partial}{\partial x} \left(-4 \frac{\partial u}{\partial t} + \frac{\partial^3 u}{\partial x^3} + 6u \frac{\partial u}{\partial x} + 3 \frac{\partial^2 u}{\partial y^2}\right) = 0,$$

are constructed as follows:\textsuperscript{20,21}

$$u(t,x,y) = 2\partial_x^2 \log \{\tau(t,x,y)\}. \quad (44)$$

The traditional $\tau$-function is given by

$$\tau_T(t,x,y) = \begin{cases} 
\sum_{i=1}^{M} \xi_M(i)e^{i\theta(t,x,y)}, & N = 1 \\
\sum_{i=1}^{M} \xi_M(i)e^{-\sum_{j=1}^{M} \theta_j(t,x,y)}, & N = M - 1 \\
\sum_{1 \leq i_1 < \ldots < i_N \leq M} \xi_M(i_1, \ldots, i_N) \left( \prod_{1 \leq j < \ell \leq N} (k_{i_j} - k_{i_\ell}) \right)^{\frac{N}{2}} e^{\sum_{j=1}^{N} \theta_j(t,x,y)}, & 2 \leq N \leq M - 2,
\end{cases} \quad (45)$$

$$k_1 < k_2 < \ldots < k_M, \quad (46)$$

$$\theta_i(t,x,y) = -k_i x + k_i^2 y - k_i^3 t. \quad (47)$$

In Eqs. (45) and (46), $M$ is the size of a set of wave numbers, $\{k_1, \ldots, k_M\}$. The sum goes over all $\binom{M}{N}$ subsets of $N$ wave numbers.

To exclude singular solutions of Eq. (44), one requires

$$\xi_M(i_1, \ldots, i_N) \geq 0. \quad (48)$$

Apart from positivity, for $N = 1$ and $N = M - 1$, the coefficients, $\xi_M(i)$, may assume arbitrary values. For $2 \leq N \leq M - 2$, $\xi_M(i_1, \ldots, i_N)$ are constrained by the Plücker relations (see, e.g., Ref. 16). For example, for $(M,N) = (4,2)$, one finds a single Plücker relation

$$\xi_4(1,2)\xi_4(3,4) - \xi_4(1,3)\xi_4(2,4) + \xi_4(1,4)\xi_4(2,3) = 0. \quad (49)$$

**A. Generating a localized source**

To generate through Eq. (9) a bounded source, $S(t,x,y)$, that is localized in the $(1 + 2)$-dimensional space, the equivalent $\tau$-function, $\tau_E(t,x,y)$, must become unbounded asymptotically in any direction in space. To achieve this, let us replace $\tau_T$ of Eq. (44) by

$$\tau_E(t,x,y) = e^{-\mu \sum_{i=1}^{M} \theta_i(t,x,y)} \tau_T(t,x,y). \quad (50)$$

$\tau_E$ of Eq. (49) is a sum of exponentials. The generic form of the exponential terms in $\tau_E$ is

$$\left( e^{-\mu \sum_{i=1}^{M} \theta_i(t,x,y) + \sum_{j=1}^{N} \theta_j(t,x,y)} \right). \quad (51)$$

For $S(t,x,y)$ to be bounded, at least some of the exponential terms must become unbounded asymptotically in any direction in the $(1 + 2)$-dimensional space. This can be achieved by requiring that the sum of the exponents in all the terms in $\tau_E$ vanishes. Then, if in any direction, some exponential terms in $\tau_E(t,x,y)$ vanish asymptotically, others become exponentially unbounded. In that sum, each $\theta_i(t,x,y)$ is multiplied by

...
Using \(\tau\) time. The position of its maximum is located at the point in the plane, for which

\[
(E - t) (M - 1) - \mu \left( \binom{M}{N} - \binom{M - 1}{N - 1} \right).
\]  

The first term in Eq. (52) counts the number of times each \(\theta_i(t,x,y)\) appears with a positive sign, and the second term counts the number of times it appears with a negative sign. As all the \(\theta_i(t,x,y)\) are independent, the vanishing of the sum requires that the coefficient of each \(\theta_i(t,x,y)\) vanishes,

\[\mu = \binom{M - 1}{N - 1}/\binom{M}{N} \] 

With this choice, \(S(t,x,y)\) of Eq. (9) generates a bounded, localized source for solutions of Eq. (43).

B. \((M \geq 2, N = 1)\)-solutions

For such solutions, Eq. (53) yields \(\mu = (1/M)\). By Eq. (45), these solutions have no wave-number dependent coefficients. Hence, usually, an \(N\)-soliton solution is reduced to an \((N - 1)\)-soliton solution when two wave numbers are made to coincide. To modify this limit, one must assign some \(\xi_M(i)\) an appropriate wave-number dependence. Making the solutions images of localized sources is possible. Here are some examples.

1. Two wave numbers: Single-soliton solution

The traditional form of the single-soliton solution is constructed from

\[\tau_Y(t,x,y) = \xi_1 \exp(\theta_1(t,x,y)) + \xi_2 \exp(\theta_2(t,x,y)).\]  

With \(M = 2, N = 1\), Eq. (53) requires \(\mu = (1/2)\), yielding

\[\tau_Y(t,x,y) = \cosh\left(\frac{1}{2}(\theta_1(t,x,y) - \theta_2(t,x,y)) + \delta\right), \quad \delta = \text{arctanh}\left(\frac{\xi_1 - \xi_2}{\xi_1 + \xi_2}\right).\]  

The resulting source to be used in Eq. (9) is

\[S(t,x,y) = \frac{1}{\cosh\left(\frac{1}{2}(\theta_1(t,x,y) - \theta_2(t,x,y)) + \delta\right)}.\]  

Thus, as in the case of the KdV equation, the single-soliton solution is the image of a single soliton given by Eq. (56). (NOT a solution of the KP II equation!)

2. Three wave numbers: Three-soliton solution \((M = 3, N = 1)\)

The traditional form of the \(\tau\)-function for the three-soliton solution \((Y\)-shaped solution) is

\[\tau_Y(t,x,y) = \xi_1 \exp(\theta_1(t,x,y)) + \xi_2 \exp(\theta_2(t,x,y)) + \xi_3 \exp(\theta_3(t,x,y)).\]  

This solution propagates rigidly in the \(x\)-\(y\) plane with a velocity given by

\[v_x = k_1k_2 + k_1k_3 + k_2k_3, \quad v_y = k_1 + k_2 + k_3.\]  

Equation (53) requires \(\mu = (1/3)\). The resulting equivalent \(\tau\)-function is

\[t E(t,x,y) = \xi_1 \exp\left(\frac{1}{3}(2\theta_1(t,x,y) - \theta_2(t,x,y) - \theta_3(t,x,y))\right) + \xi_2 \exp\left(\frac{1}{3}(2\theta_2(t,x,y) - \theta_1(t,x,y) - \theta_3(t,x,y))\right) + \xi_3 \exp\left(\frac{1}{3}(2\theta_3(t,x,y) - \theta_1(t,x,y) - \theta_2(t,x,y))\right).\]  

Using \(\tau_E\) in Eq. (9), the resulting \(S(t,x,y)\) describes a hump that is localized in the \(x\)-\(y\) plane at any time. The position of its maximum is located at the point in the plane, for which
Equation (60) yields the coordinates \( x \) and \( y \) of the point of maximum as functions of \( t \). The velocity of propagation of the source is computed to coincide with the velocity of the solution, given in Eq. (58). Figures 5 and 6 show, respectively, a three-soliton solution and its localized source.

For constant \( \xi_M(i) \), this solution tends to a single soliton solution if two of the wave numbers are made to coincide. For example, in the limit of \( k_2 \to k_1 \), it tends to a single soliton constructed out of \( k_1 \) and \( k_3 \). To make the limit vanish, one must make two \( \xi_M(i) \) vanish in that limit, making it an uninteresting option.

3. Four wave numbers: (4,1) Four-soliton solution

The traditional form for the \( \tau \)-function for the four-soliton solution with \( (M = 4, N = 1) \) is

\[
\tau_T(t,x,y) = \xi_1 \exp(\theta_1(t,x,y)) + \xi_2 \exp(\theta_2(t,x,y)) + \xi_3 \exp(\theta_3(t,x,y)) + \xi_4 \exp(\theta_4(t,x,y)).
\]  

To obtain a localized \( S(t,x,y) \), Eq. (53) requires \( \mu = (1/4) \). Figures 7 and 8 show, respectively, a four-soliton solution and its localized source, \( S(t,x,y) \).

C. Four wave numbers: (4,2) four-soliton solution

In \( (M,N) \) solutions with \( N > 1 \), the numerical coefficients depend on the wave numbers [see Eq. (45)], allowing for a dependence of the characteristics of the solutions on the wave numbers. An
FIG. 7. KP II (4,1)-soliton solution $[\tau_T$ of Eq. (61)], $k_1 = 0.1; k_2 = 0.2; k_3 = 0.3; k_4 = 0.4; \xi_1 = \xi_2 = \xi_3 = \xi_4 = 1$.

An elegant algorithm for the procedure has been found for solutions with $(M = 2k, N = k \geq 1)$, for which Eq. (43) yields $\mu = (1/2)$. The case of the (4,2) solution is discussed as an example. The traditional form of the $\tau$-function is

$$
\tau_T = \xi_4(1,2)(k_2 - k_1)e^{\theta_1 + \theta_2} + \xi_4(1,3)(k_3 - k_1)e^{\theta_1 + \theta_3} + \xi_4(1,4)(k_4 - k_1)e^{\theta_1 + \theta_4}
$$

$$
\xi_4(2,3)(k_3 - k_2)e^{\theta_2 + \theta_3} + \xi_4(2,4)(k_4 - k_2)e^{\theta_2 + \theta_4} + \xi_4(3,4)(k_4 - k_3)e^{\theta_3 + \theta_4}.
$$

In Eq. (62), $(t,x,y)$ have been omitted from $\theta_i$ for the sake of brevity. The $\xi$’s obey Eq. (49).

1. **Soliton trajectory shifts as solution parameters**

    Equation (52) requires $\mu = (1/2)$. Regrouping terms, the resulting equivalent $\tau$-function is rewritten as

$$
\tau_E(t,x,y) = \sqrt{(k_2 - k_1)(k_4 - k_3)}\xi_4(2,3)\xi_4(3,4) \cosh \left\{ \frac{1}{2} \theta_{++--} + \alpha_{++--} \right\} + \\
\sqrt{(k_3 - k_1)(k_4 - k_2)}\xi_4(1,3)\xi_4(2,4) \cosh \left\{ \frac{1}{2} \theta_{+-+--} + \alpha_{+-+--} \right\} + \\
\sqrt{(k_3 - k_2)(k_4 - k_1)}\xi_4(1,4)\xi_4(2,3) \cosh \left\{ \frac{1}{2} \theta_{+-+-} + \alpha_{+-+-} \right\}.
$$

In Eq. (63), $(t,x,y)$ have been omitted from $\theta_i$ for the sake of brevity. The $\xi$’s obey Eq. (49).

FIG. 8. $S(t,x,y)$ [Eq. (9)], the source of the (4,1)-KP II-soliton solution. The parameters are the same as in Fig. 7.
FIG. 9. KP II (4,2)-soliton solution [τE of Eq. (63)], k1 = 0.1; k2 = 0.3; k3 = 0.6; k4 = 0.9; \( \xi_{12} = 1/6; \xi_{13} = 4/15; \xi_{14} = 1/3; \xi_{23} = 1/10; \xi_{24} = 1/6; \xi_{34} = 1/15; \alpha_{+--+} = \alpha_{+---} = \alpha_{++++} = 0. \\

\[ \theta_{\sigma_1 \sigma_2 \sigma_3 \sigma_4} = \sigma_1 \theta_1 + \sigma_2 \theta_2 + \sigma_3 \theta_3 + \sigma_4 \theta_4, \]  
(64)  

\[ \alpha_{+--+} = \log \left\{ \frac{(k_2 - k_1)\xi_4(1,2)}{(k_4 - k_3)\xi_4(3,4)} \right\}, \quad \alpha_{+---} = \log \left\{ \frac{(k_3 - k_1)\xi_4(1,3)}{(k_4 - k_2)\xi_4(2,4)} \right\}, \]  
(65)  

\[ \alpha_{+++--} = \log \left\{ \frac{(k_4 - k_1)\xi_4(1,4)}{(k_3 - k_2)\xi_4(2,3)} \right\}. \]  
(65)  

(The notation is as in the case of the KdV equation.)

2. Limit of \( k_2 \rightarrow k_1 \)

Consider now the limit when two wave numbers coincide, say, \( k_2 \rightarrow k_1 \). From \( \tau_E \) of Eq. (62), one deduces that if the shifts, \( \xi_i \), are independent of the wave numbers, then the (4,2) solution is reduced to a (3,2) solution, which is a three-soliton solution (Y-shaped) with wave numbers \( k_1, k_3, \) and \( k_4 \). Constructing the solution through \( \tau_E \) of Eq. (63), this limit is reached if \( \alpha_{\sigma} \) are assigned the required singular expressions of Eq. (65). If they have other values, then the \( k_2 \rightarrow k_1 \) limit may be different. For constant \( \alpha_{\sigma} \), \( \tau_E \) of Eq. (63) tends to

\[ \tau_E(t,x,y)|_{k_2=k_1} = \sqrt{(k_3 - k_1)(k_4 - k_1)} \times \]  

\[ \left( \sqrt{\xi_4(1,3)\xi_4(2,4)} \cosh \left\{ \frac{1}{2} \left[ -(k_3 - k_4)x + \left( k_3^2 - k_4^2 \right)y - \left( k_3^3 - k_4^3 \right)t \right] + \alpha_{+--+} \right\} \right)^{\frac{1}{2}}, \]  
(66)  

which generates a single-soliton solution with wave numbers \( k_3 \) and \( k_4 \).

FIG. 10. \( S(t,x,y) \) [Eq. (9)], the source of the (4,2)-KP II-soliton solution. The parameters are the same as in Fig. 9.
3. Localized source

Using Eq. (62) in Eq. (9) yields a source, \( S(t,x,y) \), which is bounded in the (1 + 2)-dimensional space and is localized in the \( x-y \) plane at any time. The (4,2) solution is the image of this localized source under Eq. (3). Figures 9 and 10 present a (4,2) solution and its source, \( S(t,x,y) \), respectively.

IV. THE MODIFIED KdV EQUATION

The soliton solutions of the modified KdV (mKdV) equation,

\[ u_t = 6u^2u_x + u_{xxx}, \quad (67) \]

are constructed through a transformation of a different structure

\[ u(t,x) = 2\partial_x \arctan(\tau(t,x)). \quad (68) \]

Owing to the fact that the connection between the solution and the \( \tau \)-function is not through a logarithmic transformation, a simple procedure of the type described in Secs. II and III has not been found. However, the Miura transformation connecting the solutions of the KdV and mKdV equations ensures that equivalent \( \tau \)-functions can be found. Rather than embarking upon a full analysis, the case of the two- and three-soliton solutions is presented as examples.

The traditional two-soliton \( \tau \)-function is given by

\[ \tau_T(t,x) = e^{\theta_1} + e^{\theta_2} - \left( \frac{k_2-k_1}{k_1+k_2} \right)^2 e^{\theta_1} e^{\theta_2} \quad \left( \theta_i = k_i x + k_i^3 t + \delta_i \right). \quad (69) \]

\( \tau_T \) of Eq. (69) is unbounded in the vicinity of a line in the \( x-t \) plane. This singular behavior is of no concern, as it is remedied by the transformation in Eq. (68).

An equivalent \( \tau \)-function is obtained by multiplying the top and the bottom of Eq. (69) by

\[ (k_1 + k_2)^2 e^{-\frac{1}{2}(\theta_1 + \theta_2)}. \quad (70) \]

The result leads to

\[ \tau_E(t,x) = \frac{k_1 + k_2}{k_2 - k_1} \frac{\cosh \left( \frac{1}{2}(\theta_1 - \theta_2) \right)}{\sinh \left( \frac{1}{2}(\theta_1 + \theta_2) + \alpha \right)}. \quad (71) \]

The structure of \( \tau_T \) of Eq. (69) dictates the wave-number dependence of \( \alpha \) to be

\[ \sinh \alpha = -\frac{2k_1 k_2}{k_2^2 - k_1^2}. \quad (72) \]

As evident from Eq. (69), in the traditional case, the two soliton solution is reduced to a single-soliton solution when \( k_2 \to k_1 \), provided the shifts, \( \delta_i \), are constant. Using Eq. (71), this limit is a consequence of the singular nature of the traditional value of \( \alpha \), given by Eq. (72). The limit is different if \( \alpha \) does not have that singular structure. For example, for constant \( \alpha \), the leading singular term in Eq. (71) is

\[ \tau(t,x)|_{k_2 \to k_1} \sim -\frac{2k_1}{k_2 - k_1} \frac{\cosh \left( \frac{1}{2}(\delta_1 - \delta_2) \right)}{\sinh \left( k_1 x + k_1^3 t + \frac{\delta_1 + \delta_2}{2} + \alpha \right)} \quad \left( k_2 \to k_1 \right). \quad (73) \]

where the final sign depends on which side of the line

\[ k_1 x + k_1^3 t + \frac{\delta_1 + \delta_2}{2} + \alpha = 0 \quad (74) \]

is. Hence, in the limit, the \( \arctanh \) in Eq. (67) jumps between \(-\pi/2\) and \(+\pi/2\). Consequently, the limit of the two-soliton solution is a zero-width single soliton,
\[ u(t, x) \rightarrow 2\pi \delta \left( k_1 x + k_1^3 t + \frac{\delta_1 + \delta_2}{2} + \alpha \right). \quad (75) \]

The traditional three-soliton \( \tau \)-function is given by
\[ \tau_\tau(t, x) = \frac{e^{\theta_1} + e^{\theta_2} - \left( \frac{k_2 - k_1}{k_1 + k_2} \right)^2 \left( \frac{k_1 - k_1}{k_1 + k_3} \right)^2 \left( \frac{k_3 - k_2}{k_2 + k_3} \right)^2 e^{\theta_1} e^{\theta_2} e^{\theta_3}}{1 - \left( \frac{k_2 - k_1}{k_1 + k_2} \right)^2 e^{\theta_1} e^{\theta_2} - \left( \frac{k_1 - k_1}{k_1 + k_3} \right)^2 e^{\theta_1} e^{\theta_3} - \left( \frac{k_3 - k_2}{k_2 + k_3} \right)^2 e^{\theta_2} e^{\theta_3}}. \quad (76) \]

In the limit of \( k_2 \to k_1 \), Eq. (76) yields a \( \tau \)-function for a two-soliton solution with wave numbers \( k_3 \) and \( k_1 \). To see other possibilities, it pays to consider the following equivalent \( \tau \)-function:
\[ \tau_\text{E}(t, x) = \frac{\cosh((\theta_1 - \theta_2)/2) - e^{\theta_1} \left( \frac{k_1 - k_1}{k_1 + k_2} \right) \left( \frac{k_1 - k_2}{k_1 + k_3} \right) \sinh((\theta_1 + \theta_2)/2 + \beta)}{-\left( \frac{k_2 - k_1}{k_1 + k_1} \right) \sinh((\theta_1 + \theta_2)/2 + \alpha) - e^{\theta_1} \left( \frac{k_1 - k_1}{k_1 + k_2} \right) \left( \frac{k_1 - k_2}{k_1 + k_3} \right) \sinh((\theta_1 - \theta_2)/2 + \gamma)}. \quad (77) \]

The traditional values for the shifts are
\[ \sinh \alpha = \frac{2k_1 k_2}{k_2^2 - k_1^2}, \quad (78) \]
\[ \sinh \beta = \frac{2 \left( k_1^2 k_2 + k_2^2 k_2 + k_1 k_2 k_3 \right) \left( k_1 k_2^2 + k_2^2 k_3 + k_2 k_3^2 + k_1 k_2 k_3 \right)}{(k_2^2 - k_1^2) \left( k_2^2 - k_1^2 \right) \left( k_3^2 - k_2^2 \right)}, \quad (79) \]
\[ \sinh \gamma = \frac{2 \left( k_2 - k_1 \right) \left( k_3^2 - k_1 k_2 \right) k_3}{(k_3^2 - k_1^2) \left( k_3^2 - k_2^2 \right)}. \quad (80) \]

If the shifts obtain these traditional values, then their singular behavior ensures that the solution tends to a two-soliton solution with wave numbers \( k_1 \) and \( k_3 \) in the limit of \( k_2 \to k_1 \). If, however, the shifts do not have the singular nature dictated by Eqs. (78)–(80), then the solution tends to a single soliton with wave number \( k_3 \)!

V. CONCLUDING COMMENTS

In this paper, it has been shown that the well-known non-uniqueness of the \( \tau \)-functions, from which soliton solutions of nonlinear evolution equations are constructed, allows for the association of soliton solutions with bounded and spatially localized sources. This has been achieved through the construction of \( \tau \)-functions that are equivalent to the form used traditionally. The physics gain is the fact that all the information regarding the structure if a multi-soliton solution is contained in a source that is localized in the soliton interaction region.

A byproduct of this study is the observation that the equivalent \( \tau \)-functions allow for a simple exposition of the role of shifts in the positions of soliton trajectories in determining solution characteristics. In particular, the limit into which an \( N \)-soliton solution is reduced when two wave numbers are made to coincide has been elucidated. In the case of the KdV equation, this allows for seeing in a simple manner when an \( N \)-soliton solution degenerates into a traditional \((N-1)\)-soliton solution and when into an \((N-2)\)-soliton solution. In the case of the KP II equation, the discussion of the \((4,2)\) four-soliton solution shows when it degenerates into a three-soliton solution and when to a single soliton. In the case of the mKdV equation, depending on the shift, a two-soliton solution is reduced to a single-soliton solution or may degenerate into a \( \delta \)-function.

Clearly, the analysis presented here can be applied to other KdV-like equations, such as the bi-directional KdV equation\(^{25}\) and the Sawada-Kotera\(^{26}\) equation.

APPENDIX: CONSTRUCTION OF EQUIVALENT \( \tau \)-FUNCTION FOR \( N \)-SOLITON SOLUTION OF KdV EQUATION

One first multiplies the traditional \( \tau \)-function,
\[ \tau_T(t,x) = 1 + \sum_{n=1}^{N} \sum_{1 \leq i_1 < i_2 < \cdots < i_n \leq N} \left( \prod_{j=1}^{n} \prod_{l=j+1}^{N} \frac{(k_{i_l} - k_{i_j})}{k_{i_l} + k_{i_j}} \right)^2 e^{2 \sum_{m=1}^{N} \theta_m}, \]  

(A1)

by

\[ \left( \prod_{j=1}^{N} \prod_{l=j+1}^{N} (k_j + k_l)^2 \right) \prod_{i=1}^{N} e^{-\theta_i}. \]  

(A2)

The \(2^N\) exponential terms in the result,

\[ \left( \prod_{j=1}^{N} \prod_{l=j+1}^{N} (k_j + k_l)^2 \right) \prod_{i=1}^{N} e^{-\theta_i} \left( 1 + \sum_{n=1}^{N} \sum_{1 \leq i_1 < i_2 < \cdots < i_n \leq N} \left( \prod_{j=1}^{n} \prod_{l=j+1}^{N} \frac{(k_{i_l} - k_{i_j})}{k_{i_l} + k_{i_j}} \right)^2 e^{2 \sum_{m=1}^{N} \theta_m} \right), \]  

(A3)

are split into \(2^{N-1}\) pairs of terms. The simplest pair is the one in which all \(\theta_i\) have the same signs. It is obtained from the sum of the following two terms in \(\tau_T\):

\[ 1, \left( \prod_{j=1}^{N} \prod_{l=j+1}^{N} \frac{(k_{i_l} - k_{i_j})}{k_{i_l} + k_{i_j}} \right)^2 e^{2 \sum_{m=1}^{N} \theta_m}. \]  

(A4)

When these are multiplied by the factor of Eq. (A2), the sum of the two terms becomes

\[ \left( \prod_{j=1}^{N} \prod_{l=j+1}^{N} (k_j + k_l)^2 \right) e^{-\sum_{i=1}^{N} \theta_i} + \left( \prod_{j=1}^{N} \prod_{l=j+1}^{N} (k_j - k_l)^2 \right) \sum_{i=1}^{N} \theta_i = \]

\[ \left\{ \prod_{j=1}^{N} \prod_{l=j+1}^{N} (k_j + k_l)^2 + \prod_{j=1}^{N} \prod_{l=j+1}^{N} (k_j - k_l)^2 \right\} \cos \theta_{++\cdots+}^{N,\text{N times}} + \]

\[ \left\{ \prod_{j=1}^{N} \prod_{l=j+1}^{N} (k_j - k_l)^2 - \prod_{j=1}^{N} \prod_{l=j+1}^{N} (k_j + k_l)^2 \right\} \sin \theta_{++\cdots+}^{N,\text{N times}}. \]  

(A5)

This term can be re-written as

\[ \left( 2 \prod_{j=1}^{N} \prod_{l=j+1}^{N} (k_j + k_l) \right) \left( \prod_{j=1}^{N} \prod_{l=j+1}^{N} (k_j - k_l) \right) \cos \theta_{++\cdots+}^{N,\text{N times}} + \alpha_{++\cdots+}^{N,\text{N times}}, \]  

(A6)

where

\[ \sinh \alpha_{++\cdots+}^{N,\text{N times}} = \frac{\prod_{j=1}^{N} \prod_{l=j+1}^{N} (k_j - k_l)^2 - \prod_{j=1}^{N} \prod_{l=j+1}^{N} (k_j + k_l)^2}{2 \prod_{j=1}^{N} \prod_{l=j+1}^{N} (k_j + k_l) \prod_{j=1}^{N} \prod_{l=j+1}^{N} (k_j - k_l)}. \]  

(A7)

The pair of the next level of complication is that in which one of the \(\theta_i\) has a negative sign. Take as an example the case that this is \(\theta_N\). It is obtained from the sum of the following two terms in \(\tau_T\):

\[ e^{2\theta_N}, \left( \prod_{j=1}^{N-1} \prod_{l=j+1}^{N-1} \frac{(k_{i_l} - k_{i_j})}{k_{i_l} + k_{i_j}} \right)^2 e^{2 \sum_{m=1}^{N-1} \theta_m}. \]  

(A8)
When these are multiplied by the factor of Eq. (A2), the sum of the two terms becomes

\[
\left\{ \prod_{j=1}^{N} \prod_{l=j+1}^{N} (k_j + k_l)^2 \right\} e^{(-N\sum_{i=1}^{N} \theta_i) + \theta_N} + \prod_{j=1}^{N-1} (k_N + k_j) \left( \prod_{j=1}^{N-1} \prod_{l=j+1}^{N} (k_j - k_l) \right) e^{(-N\sum_{i=1}^{N} \theta_i) - \theta_N} =
\]

\[
\left\{ \prod_{j=1}^{N} \prod_{l=j+1}^{N} (k_j + k_l)^2 \right\} e^{(-N\sum_{i=1}^{N} \theta_i)} + \prod_{j=1}^{N-1} (k_N + k_j) \left( \prod_{j=1}^{N-1} \prod_{l=j+1}^{N} (k_j - k_l) \right) \cos \theta_{++\cdots+} +
\]

(A9)

This term can be re-written as

\[
\left( 2 \prod_{j=1}^{N} \prod_{l=j+1}^{N} (k_j + k_l) \right) \prod_{l=j+1}^{N} (k_N + k_l) \left( \prod_{j=1}^{N-1} \prod_{l=j+1}^{N} (k_j - k_l) \right) \cos \left( \theta_{++\cdots+} + \alpha_{++\cdots+} \right),
\]

(A10)

where

\[
\sin \alpha_{++\cdots+} = \frac{N-1 \prod_{j=1}^{N-1} (k_N + k_j)^2 \prod_{j=1}^{N-1} \prod_{l=j+1}^{N} (k_j - k_l) - \prod_{j=1}^{N-1} \prod_{l=j+1}^{N} (k_j + k_l)^2}{2 \prod_{j=1}^{N} \prod_{l=j+1}^{N} (k_j + k_l) \prod_{j=1}^{N-1} \prod_{l=j+1}^{N} (k_j - k_l)}.
\]

(A11)

The next level of complication is of pairs, in which two of the \( \theta_i \) have a negative sign. Take as an example the case that these are \( \theta_N \) and \( \theta_{N-1} \). It is obtained from the sum of the following two terms in \( \mathcal{T}_\tau \):

\[
e^{2(\theta_{N-1}+\theta_N)} \left( \prod_{j=1}^{N-2} \prod_{l=j+1}^{N-2} \left( k_{ij} + k_{lj} \right)^2 \right) e^{2\sum_{m=1}^{N-2} \theta_m}.
\]

(A12)

When these are multiplied by the factor of Eq. (A2), their sum becomes

\[
\left\{ \prod_{j=1}^{N} \prod_{l=j+1}^{N} (k_j + k_l)^2 \right\} e^{(-N\sum_{i=1}^{N} \theta_i) + \theta_N} + \prod_{j=1}^{N-1} (k_N + k_j) \left( \prod_{j=1}^{N-1} \prod_{l=j+1}^{N} (k_j - k_l) \right) e^{(-N\sum_{i=1}^{N} \theta_i) - \theta_N} =
\]

\[
\left\{ \prod_{j=1}^{N} \prod_{l=j+1}^{N} (k_j + k_l)^2 \right\} e^{(-N\sum_{i=1}^{N} \theta_i)} + \prod_{j=1}^{N-1} (k_N + k_j) \left( \prod_{j=1}^{N-1} \prod_{l=j+1}^{N} (k_j - k_l) \right) \cos \theta_{++\cdots+} +
\]

(A13)

This term can be re-written as

\[
\left( 2 \prod_{j=1}^{N} \prod_{l=j+1}^{N} (k_j + k_l) \right) \prod_{l=j+1}^{N} (k_N + k_l) \left( \prod_{j=1}^{N-1} \prod_{l=j+1}^{N} (k_j - k_l) \right) \cos \left( \theta_{++\cdots+} + \alpha_{++\cdots+} \right),
\]

(A14)
where
\[
\sinh \alpha = \frac{N^{-1} \prod_{i=1}^{N-1} (k_N + k_i)^2 \prod_{i=1}^{N-2} (k_{N-1} + k_i)^2 \left( \prod_{j=1}^{N-2} \frac{N-2}{i=j+1} \left( k_j - k_i \right)^2 \right) - N \prod_{j=1}^{N} \prod_{l=j+1}^{N} \left( k_j + k_l \right)^2}{\left( 2 \prod_{j=1}^{N} \prod_{l=j+1}^{N} \left( k_j + k_l \right) \right)^2 \prod_{j=1}^{N} \prod_{l=j+1}^{N} \left( k_j - k_l \right)^2}.
\] 

(A15)

The construction of the remaining part of \( \tau_E \) follows similar steps. All other pairs of exponential terms in \( \tau_E \) are treated in a similar manner. Note that in the three examples discussed above, the final form of each contribution is proportional to a constant multiplicative factor
\[
2 \prod_{j=1}^{N} \prod_{l=j+1}^{N} \left( k_j + k_l \right).
\] 

(A16)

As any overall constant multiplicative factor in a \( \tau \)-function does not affect \( u(t,x) \) given by Eq. (2), for aesthetic reasons, in the definition of the equivalent \( \tau \)-function, \( \tau_E \), let us divide each term by
\[
2 \prod_{j=1}^{N} \prod_{l=j+1}^{N} \left( k_j + k_l \right)^2.
\] 

(A17)

As a result, the contributions to \( \tau_E \) of the three terms discussed in detail above become
\[
\text{Eq. (A6): } \left( \prod_{j=1}^{N} \prod_{l=j+1}^{N} \frac{k_j - k_l}{k_j + k_l} \right) \cos \left( \theta_{N-\text{times}} + \alpha_{N-\text{times}} \right),
\] 

(A18)

\[
\text{Eq. (A10): } \left( \prod_{j=1}^{N-1} \prod_{l=j+1}^{N-1} \frac{k_j - k_l}{k_j + k_l} \right) \cos \left( \theta_{N-\text{times}} + \alpha_{N-\text{times}} \right),
\] 

(A19)

\[
\text{Eq. (A14): } \left( \prod_{j=1}^{N-2} \prod_{l=j+1}^{N-2} \frac{k_j - k_l}{k_j + k_l} \right) \cos \left( \theta_{N-2\text{times}} + \alpha_{N-2\text{times}} \right).
\] 

(A20)

In each term, the multiplicative wave number dependent coefficients
\[
\frac{k_j - k_l}{k_j + k_l}
\] 

(A21)

appear only for indices \( i \) and \( j \), for which
\[
\sigma_j = \sigma_i.
\] 

(A22)

This pattern recurs in all other terms. Hence, the general term in \( \tau_E \) has the form
\[
\left( \prod_{j=1}^{N} \prod_{l=j+1}^{N} \frac{k_j - k_l}{k_j + k_l} \right) \cos(\theta_{\tilde{\sigma}} + \alpha_{\tilde{\sigma}}),
\] 

(A23)

where
\[
(\tilde{\sigma} = (\sigma_1, \sigma_2, \ldots, \sigma_N), \quad \sigma_1 = \pm 1, \sigma_{i \geq 2} = \pm 1).
\] 

(A24)

Next, note that the same denominator appears in the definitions of all \( \alpha_{\tilde{\sigma}} \). Thus, in the traditional (inverse-scattering/Hirota) construction, the shifts in the position of soliton trajectories in the \( x-t \) plane all have a singular dependence on the wave numbers whenever any two of them coincide.
Finally, in the traditional construction, the exponents, $\theta_i$, may include arbitrary shifts, $\delta_i$,

$$\theta_i = k_i x + \omega_i t + \delta_i. \quad (A25)$$

The cumulative contribution of these shifts in any term,

$$\theta_{\vec{\sigma}} = \sum_{i=1}^{N} \sigma_i \theta_i, \quad (A26)$$

is

$$\sum_{i=1}^{N} \sigma_i \delta_i. \quad (A27)$$

In $\tau_E$, the $2^{N-1} \alpha_{\vec{\sigma}}$ constitute additional shifts in the locations of soliton trajectories in the $x$-$t$ plane. Hence, they should be in a similar form

$$\alpha_{\vec{\sigma}}(\vec{k}) = \sum_{i=1}^{N} \sigma_i \alpha_i(\vec{k}), \quad (\vec{k} = \{k_1, k_2, \ldots, k_N\}). \quad (A28)$$

This ensures translation invariance along the trajectory of each soliton, once sufficiently far away from all other solitons. This allows for re-writing the equivalent $\tau$-function, $\tau_E$, in the form

$$\tau_E(t, x) = \prod_{\vec{\sigma}} \prod_{i=1}^{N} \prod_{j=1}^{N} \frac{(k_j - k_i)}{(k_j + k_i)} \cosh \tilde{\theta}_{\vec{\sigma}}, \quad (A29)$$

$$\tilde{\theta}_{\vec{\sigma}} = \sum_{i=1}^{N} \sigma_i \tilde{\theta}_i, \quad (A29)$$

$$\tilde{\theta}_i = \theta_i + \alpha_i(\vec{k}) = k_i x + \omega_i t + \Delta_i(\vec{k}), \quad \Delta_i(\vec{k}) = \delta_i + \alpha_i(\vec{k}).$$

The shifts, $\Delta_i(\vec{k})$, may contain the wave number dependent contributions required in the traditional case but, as discussed in the main body of the paper, may assume any values.