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On the virtual cathode oscillator’s energy optimization

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This paper presents a method for the maximization of the energy radiated by a Vircator at a given band of frequencies. The research focuses on axially extracted Vircators. The methodology carried out uses a combination of analytic and numerical techniques. The solution presented is useful in cases where the Vircator is available and drastic changes could be burdensome. Results are validated by numerical simulations. © 2018 Author(s). All article content, except where otherwise noted, is licensed under a Creative Commons Attribution (CC BY) license (http://creativecommons.org/licenses/by/4.0/).

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I. INTRODUCTION

Virtual Cathode Oscillator (Vircator) is a type of narrowband High-power Microwaves source (HPMS). With regard to other type of HPMS, Vircators present some advantages, such as smaller size, reduced weight, lower cost, and its operation does not require the use of external magnetic fields.1,2 However, Vircators primary drawback is its low efficiency.

The overall geometry of the device is depicted in Fig. 1. The Vircator is formed by a vacuum chamber where two regions are identified. The first region is the diode, which is comprised of a cathode of radius \( r_c \) and an anode of transparency \( T_a \). Cathode and anode are spaced a distance \( d \). The second region is the drift-tube, which presents a radius \( r_{dt} \) and maximum drift current \( I_{scl} \). When the current injected into the drift-tube exceeds \( I_{scl} \), it appears a space area where the charge accumulates. This region is known as the Virtual Cathode (VC). The oscillation of the VC and the electrons reflected from the VC to the cathode are the two phenomena responsible for the microwaves generation.3,4

We are interested in determining the set of conditions leading to a maximization on the energy generated by axially-extracted Vircators at a given frequency.

The set of parameters one could vary in order to improve the efficiency, define the Vircator geometry and its point of operation. These are: the anode-cathode voltage \( (V) \), cathode radius \( (r_c) \), anode-cathode gap \( (d) \), anode transparency \( (T_a) \) and drift-tube radius \( (r_{dt}) \) (see Fig. 1). However, in this paper we will focus only on \( V, r_c \) and \( d \), while \( r_{dt} \) is considered constant (due to physical constrains). The parameter \( T_a \) is generally chosen based on the desired anode lifetime of the anode, and will also be considered constant in this paper.

The paper is organized as follows: In Section II, the theoretical problem is formulated. Section III presents the energy efficiency optimization. Validation by computational simulation is presented in Section IV. Finally, conclusions are given in Section V.

II. PROBLEM FORMULATION

The optimality criterion, for the case, will be defined as the maximization of the energy \( (E) \), produced at one specific frequency band. \( E \) can be defined from the frequency domain expressions as:
FIG. 1. Scheme of a Vircator axially extracted.

\[ E = \int_{f_l}^{f_h} ESD(f) df \]  

where \( f_l \) and \( f_h \) are the limit frequencies of the band of interest and \( ESD(f) \) is the energy spectral density available at the extraction window of the device.

On the other hand, if during a time \( t_\omega = t_f - t_i \), the Vircator radiates in the desired band, \( E \) can be calculated from the time domain expression:

\[ E = \int_{t_i}^{t_f} P(t) dt \]  

where \( t_i \) and \( t_f \) are the initial and final times of the radiation burst, and \( P(t) \) is the instantaneous power radiated into the desired band. Being \( P(t) \) the key term of the optimization problem, well discuss it in detail, as follows:

A. Time-invariant definition of the instantaneous power in a vircator

Vircators produce radiation around two dominant frequencies.\(^3,4\) The first frequency \( (f_r) \) is relates to the electrons reflected between the cathode and the VC. The second frequency \( (f_{vc}) \) is due to the oscillation of the VC. Typically, the power radiated by the VC oscillation is higher than the produced by the electron reflection.\(^5\) For this reason, we focus only on the radiation generated by the VC oscillation.

The first attempt to define \( P(t) \) is a time-invariant model based on the Larmor’s formula.\(^5–7\) This approach calculates the power radiated by the VC oscillation as the instantaneous power radiated by an accelerated, equivalent punctual charge \( (\dot{Q}) \) as follows

\[ P(t) = \frac{Q^2 a(t)^2}{6\pi\varepsilon_0 c^3} = \frac{Q^2 x_p^2 \omega_{vc}^4 \cos^2(\omega_{vc} t)}{6\pi\varepsilon_0 c^3} \]  

where \( Q \) is the accumulated charge into the VC, \( a(t) = x_p \omega_{vc}^2 \cos(\omega t) \) is the instantaneous acceleration of the charge, considering simple harmonic motion, \( c \) is the speed of light, \( \varepsilon_0 \) is the free space permittivity, \( x_p \) is the maximum displacement of the VC respect to its mean position and \( \omega_{vc} = 2f_{vc} \) is the oscillation angular velocity.

Equation (3) gives the instantaneous power. Furthermore, the energy radiated at the frequency \( \omega_{vc} \) can be obtained as the product of the average power \( (\bar{P}) \) and \( t_\omega \) (time during which the Vircator radiates only at the frequency \( \omega_{vc} \)). \( \bar{P} \) therefore, is given by the following expression:

\[ \bar{P} = \frac{1}{T} \int_0^T \frac{Q^2 \omega_{vc}^4 x_p^2 \cos^2(\omega_{vc} t)}{6\pi\varepsilon_0 c^3} dt = \frac{Q^2 \omega_{vc}^4 x_p^2}{12\pi\varepsilon_0 c^3} \]  

B. Effect of the time-varying charge in the VC power

The model presented in Equation (4) fails to predict accurately the radiated power, as it does not consider two important issues:

In the first place, the charge acceleration \( (a(t)) \) is not simply harmonic (as it is shown by J. Benford et al.,\(^3\) Fig. 10.1). This circumstance produces spectral dispersion and introduces a magnitude error.
in the average power predicted by Equation (4). This dispersion will be treated here by introducing a multiplicative correction factor \( F_1 = \epsilon_a \).

On the other hand, the model proposed in Equation (4) does not take into account the fact that \( Q \) is time-dependent. In order to adjust the model to the time domain dependency of the charge in the virtual cathode, we introduce a second correction factor \( F_2 = \bar{Q}_2(\gamma_0^2 - 1)/Q^2 \), where \( \gamma_0 \) is the relativistic factor of the electrons at the anode and:

\[
\bar{Q} \equiv \frac{I_b}{\omega_{vc}} e^{-2\left(1 - \frac{I_{sc}}{I_b}\right)^2},
\]

(5)

where \( I_b \) is drift-tube injected current, and \( I_{sc} \) is the Space-Charge-Limiting current.

This term was obtained through a process of parametric variation using a 2.5D Particle In Cell (PIC) simulation using the software XOOPIC\(^1\) and a mathematical fitting procedure. Notice that both \( F_1 \) and \( F_2 \) are time independent.

Using the two correction factors and the definition of \( x_p \) reported in Ref. 7, we can redefine the following expression for the average power radiated by the equivalent VC time varying-charge

\[
\bar{P} \equiv \frac{\pi^2}{24\epsilon_0 mc^2} \omega_{vc}^2 I_b(\gamma_0^2 - 1)e^{-4\left(1 - \frac{I_{sc}}{I_b}\right)^2} \epsilon_a,
\]

(6)

where \( r_b \) is the drift-tube beam radius.

C. Definition of the objective function

Equation (6) is the objective function to optimize. It’s adequate at this point to rewrite it as a function of the parameters defining the geometry and the operating point of the Vircator. In order to do this, it is necessary to make some transformations according to the following considerations:

1. The relativistic plasma frequency (\( f_p = \omega_p/2\pi \)) can be stated as \(^3\)

\[
f_p = \frac{1}{2\pi} \left( \frac{n_b e^2}{\epsilon_0 m \gamma_0} \right)^{1/2},
\]

(7)

where \( n_b \) is the drift-tube injected electrons density and can be defined as \(^4\)

\[
n_b = \frac{J_b}{e v_0},
\]

(8)

where \( J_b \) is the beam density current and \( v_0 \) is the anode electrons velocity given by

\[
v_0 = \frac{c}{\gamma_0} \sqrt{\gamma_0^2 - 1}.
\]

(9)

On the other hand, \( \omega_{vc} \) can be defined as \( a_1 \omega_p \). And the coefficient \( 1 < a_1 < 2.5 \). \(^1\)

Replacing Eqs. (8) to (9) into (7), it is obtained

\[
\omega_{vc} = a_1 \omega_p = a_1 \sqrt{\frac{e J_b}{\epsilon_0 m c \sqrt{\gamma_0^2 - 1}}},
\]

(10)

2. \( \gamma_0 \) relates to the anode-cathode voltage (\( V \)) through the energy conservation law as:

\[
\gamma_0 = \frac{e V}{m c^2} + 1,
\]

(11)

where \( m \) and \( e \) are the electron mass and charge.

3. For the axial geometry here considered \( I_{sc} \) is defined as:\(^3\)

\[
I_{sc} = \frac{2\pi \epsilon_0 m c^3}{e} \left( \frac{\gamma_0^{2/3}}{3} - 1 \right)^{3/2} \ln(r_{vb}/r_b).
\]

(12)
4. The beam current \((I_b)\) relates to the diode current \((I_d)\) and the anode transparency \((T_a)\) as:

\[
I_b = I_d T_a .
\]  
\[\text{(13)}\]

5. If one-dimensional electron flow is considered, the following approximation holds:

\[
r_b = r_c .
\]  
\[\text{(14)}\]

6. The current \(I_b\) is defined as:

\[
I_b = \pi r_b^2 J_b .
\]  
\[\text{(15)}\]

7. Finally, the diode density current \((J_d)\) can be approximated with the Child-Langmuir’s Law\(^{12,13}\) as

\[
J_{CL} = \frac{4}{9} \varepsilon_0 \sqrt{\frac{2e V^{3/2}}{m d^2}} .
\]  
\[\text{(16)}\]

or the relativistic expression\(^{14,15}\) as

\[
J_d = J_{CL} 2F_1 \left( \frac{1}{4}; \frac{3}{4}; \frac{7}{4}; -\frac{eV}{2mc^2} \right)^2 ,
\]  
\[\text{(17)}\]

where \(2F_1()\) is the hypergeometric function.\(^{16}\) Equation (17) is the compact form of the exact relativistic solution for the Space-Charge-Limited current of planar diodes and is equivalent to the classical solution presented by R. Jory and A.W. Trivelpiece.\(^{17}\)

Taken into account these considerations, the objective function to optimize, this is, the average power, can be directly written as a function of \(V, r_c,\) and \(r_{dt}\), as follows:

\[
\bar{P} = \frac{\pi^3 a^2 m^2}{12 e^2} \omega_p^2 \left( \frac{eV + 1)^2 - 1}{1 - \frac{1}{\sqrt{\gamma_0^2 - 1}}} \right)^{3/2} ,
\]  
\[\text{(18)}\]

where \(k = e/(mc^2)\).

Validity of this model will be analyzed in the Section IV.

D. Constraints

As in any optimization problem, there is a set of constrains to be considered, as follows:

1. The first constraint regards Eq. (19): the electric field at the cathode surface must exceed the emission threshold, in order the electron emission process to take place. For planar geometries, the relation \(V/d\) approximates the electric field at any distance from the cathode, therefore:

\[
V/d > E_{th} .
\]  
\[\text{(19)}\]

2. The term \(I_{scf}\) (Eq. (12)) gives the second inequality constraint. To ensure the VC formation, the current injected into the drift-tube \((I_b)\) must exceed \(I_{scf}\):\(^3\)

\[
I_b > I_{scf} .
\]  
\[\text{(20)}\]

3. The diode current \((I_d)\) defines the following constraint: \(I_d\) cannot exceed the beam pinching current \((I_{pinch})\):\(^3\) Additionally, to ensure both one-dimensional and laminar electron flow (this criterion is mandatory to consider \(r_b = r_c\)), the diode current must not exceed a critical current magnitude \((I_c)\) given by\(^{18}\)

\[
I_d \leq \frac{2\pi e_0 mc^3}{e} \frac{r_c}{d} \sqrt{\gamma_0^2 - 1} .
\]  
\[\text{(21)}\]

4. Anode transparency is a value between 0 and 1 (0 for an anode totally shielded and 1 for an anode totally transparent).

\[
0 \leq T_a \leq 1 .
\]  
\[\text{(22)}\]

5. Finally, the cathode radius should be smaller than the drift-tube radius.

\[
r_c < r_{dt} .
\]  
\[\text{(23)}\]
III. OPTIMIZATION

Now that the objective function has been defined by Eq. (18) and constraints stated by Eqs. (19) to (23), we will proceed with the solution of the optimization problem, consisting, as it was said, on maximizing \( \bar{P} \) maintaining \( \omega_{vc} \) in the range defined by \( \omega_{vc} = a_{1}\omega_p \).

As a first attempt at understanding the problem, Fig. 1 presents the level curves of the average power calculated for a parametric variation of \( r_c \) and \( V \) while \( \omega_p, T_a, \) and \( r_{dt} \) remain constant. Notice that the area plotted in Fig. 2 defines the whole space of solutions for \( \bar{P} \).

However, a closed analysis of the results presented in Fig. 2 shows that the average power, stated by Eq (18), maximizes when the diode current equals the one-dimensional and laminar criterion, i.e: \( I_d = I_c \). The locus corresponding to this condition is the dashed curve highlighted in Fig. 2. This consideration simplifies the optimization problem, as the search space for the optimizing values \( (V, r_c) \), reduces from a 2-D surface to the curve defined by \( I_d = I_c \).

Using Eqs. (11), (16), (12) and (21), it can be proven that, for the condition \( I_d = I_c \) to happen, the following equation must hold:

\[
x_1(kV)^2 - kV - 2 = 0,
\]

where \( x_1 = (2r_c^4\omega_p^4)/(81c^4T_a^2) \).

Eq. (24) can be further reduced, arriving to the following expression for the curve \( I_d = I_c \):

\[
r_c = \frac{3c}{2^{1/4}\omega_p} \left( \frac{kV + 2}{\sqrt{kV}} \right)^{1/4} \sqrt{T_a}.
\]

The optimization problem consists, therefore, on finding the exact couple of values \( (V, r_c) \) satisfying Eq. (25) and maximizing \( \bar{P} \).

When the condition \( I_d = I_c \) appears, the equation for the average power \( \bar{P} \), given by Eq. (18), can be rewritten as:

\[
\bar{P}(V) = c_1 G_1(V)e^{-4(1-c_2 G_2(V))^2},
\]

where \( c_1 \) and \( c_2 \) are positive constants. \( G_1(V) \) and \( G_2(V) \) are positive and monotonic increasing functions for \( V > 255kV \) and are given by

\[
G_1(V) = \left( \frac{2 + kV}{V} \right)^{3/2},
\]

\[
G_2(V) = \frac{V(kV + 1)^{2/3} - 1}{\sqrt{kV + 2(kV + 1)^{2/3}} - 1 + \ln \left( \frac{2^{1/4}r_{dt}\omega_p\sqrt{kV}}{3c\sqrt{T_a}(kV+2)^{1/4}} \right)}.
\]

FIG. 2. Average power as function of \( V \) and \( r_c \) at fixed \( \omega_p = 2\pi f_p, f_p = 2.83 \text{GHz}, r_{dt} = 5 \text{cm} \) and \( T_a = 0.5 \). The dashed line shows the curve \( I_d = I_c \) which is the limit given by the constraint number 3.
If Eq. (26) is derived and equaled to zero, the maximum appears when:

\[ G'_2(V)(c_2G_2(V) - 1) = \frac{G'_1(V)}{8c_2G_1(V)} \]  

(29)

\[ G'_2(V)(c_2G_2(V) - 1) \] is strictly increasing function of \( V \), whereas \( G'_1(V)/(2c_2G_1(V)) \) is strictly decreasing function of \( V \). The match of these two functions occurs only one time and Eq. (26) presents only one maximum. The optimal \( V \) can, therefore, be found solving numerically Eq. (29) or using any local optimization technique such as the gradient method.

Summarizing: for a Vircator with given drift-tube radius \( (r_{dt}) \) and anode transparency \( (T_a) \), the optimal parameters maximizing the radiated energy at a given \( f \) can be found as follows

1. The first step is to define the value of \( \omega_p = 2\pi f/a_1 \), where \( a_1 \) is taken as the central value 2.12\(^{19}\) for design.
2. Optimal \( V \) is found optimizing the Eq. (18), where \( r_c \) is given by Eq. (25).
3. \( \gamma_0 \) is calculated from Eq. (11).
4. Beam current density \( (J_b) \) is solved from Eq. (10).
5. Diode current density \( (J_d) \) is calculated for laminar flow as \( J_d = J_b/T_a \).
6. Optimal \( d \) is solved from the Child-Langmuir’s law stated in Eq. (16). To obtain a higher precision, the relativistic correction can be used.\(^{14}\)

Appendix presents the proof that for any case the maximum always is located on the curve \( I_d = I_c \).

IV. VALIDATION

To validate the presented approach, we tested three different geometries, listed in Table I.

The validation is twofold. In the first case, Vircator #1 was simulated on a grid of points distributed on the whole surface presented in Fig. 2. The points where generated as a full combination of the following values:

\( V = [400 \ 800 \ 1600 \ 3200 \ 6400] \) kV and \( r_c = [1 \ 1.5 \ 2 \ 2.5 \ 3 \ 3.5] \) cm.

We selected also points on the curve \( I_d = I_c \) (see Table II for details).

The second case consisted on the simulation of Vircators #2 and #3. This validation was performed only for a set of values \( (V, r_c) \) laying on the curve \( I_d = I_c \) (see Figs. 4 and 5, and Table II).

A. Simulation setup

In all the cases, the results predicted by the model here presented were compared with the results of 2.5D PIC simulations carried out in XOOPIC\(^{10}\) (see columns 6 and 7, Table II). Each simulation was performed during 40ns on which the anode-cathode voltage \( (V) \) was hold constant at a prefixed value. The number of cells in \( \hat{r} \) was fixed to produce an exact representation of the cathode. For the Vircator #1 the number of cell in \( \hat{r} \) direction was 40, for Vircator #2 was 48, for Vircator #3 was 30. On the other hand, the number of the cells in \( \hat{z} \) direction was calculated for each simulation, in order to fit the anode-cathode gap distance, and to get an overall Vircator length of 40cm. The electron emission model used was the FieldEmmitter.\(^{20}\) The cathode length \( (L_c) \) was represented with three cells (see Fig. 1).

TABLE I. Vircators to optimize.

<table>
<thead>
<tr>
<th>Vircator</th>
<th>( f ) [GHz](^a)</th>
<th>( r_{dt} ) [cm]</th>
<th>( T_a )</th>
</tr>
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<tr>
<td>1</td>
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<tr>
<td>3</td>
<td>8</td>
<td>6</td>
<td>0.9</td>
</tr>
</tbody>
</table>

\(^a\omega = 2\pi f \) and \( \omega_p = \omega/2.12 \).
TABLE II. Simulation points, results and model predictions.

<table>
<thead>
<tr>
<th>Vircator</th>
<th>Point</th>
<th>$r_c$ [cm]</th>
<th>$V$ [kV]</th>
<th>$d$ [cm]</th>
<th>simul</th>
<th>model$^a$</th>
</tr>
</thead>
<tbody>
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<td>1</td>
<td>1</td>
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<td>56.8</td>
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<td>1</td>
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<td>0.80</td>
<td>59.4</td>
<td>0.6</td>
</tr>
<tr>
<td>3</td>
<td>1.5</td>
<td>1</td>
<td>400</td>
<td>0.62</td>
<td>54.4</td>
<td>70.7</td>
</tr>
<tr>
<td>4</td>
<td>1.5</td>
<td>1</td>
<td>800</td>
<td>0.80</td>
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<td>70.9</td>
</tr>
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$^a$It was calculated for $a_1 = 2.12$. $\epsilon_a$ was not considered.

The radiated energy was calculated from the numerically obtained fields as follows: the time-domain electric field, in direction $\hat{r}$ and magnetic field in direction $\hat{\phi}$, were recorded on each cell at the extraction surface. The corresponding frequency domain expressions ($E_f$ and $H_f$) were calculated using the Fourier transform. The power spectral density on the extraction window was obtained as:

$$ PSD = \frac{1}{2\pi} \int (E_f \times H_f^*) \, dA, \quad (30) $$

where $^*$ denote the conjugated operator and $A$ is the surface of the extraction window.

Finally, the energy radiated ($E$) on the band from $1.9f_p$ to $2.5f_p$ was calculated using Equation (1).

To validate the simulation setup, we use as benchmark the geometry of a Vircator presented by Eun-ha Choi et al.\textsuperscript{22} ($r_d = 4.8cm$, $d = 0.5cm$, $r_c = 2cm$, $T_a = 50\%$, and $V = 290kV$). The authors reported a measured dominant frequency defined between 6.68GHz and 7.19GHz. The paper also
reports results from a simulation performed using the software MAGIC, which predicts a dominant frequency of 6.7GHz. On the other hand, the PSD of our simulation using XOOPIC can be seen in Fig. 3. The dominant frequency is 6.5GHz, which means that there is a deviation of 0.2GHz or 3% respect to the benchmark simulation.

B. Results

Table II compares the results of each set of simulations with the prediction of the model (Eq. (18)). The Points where the maximum average power was obtained are highlighted.

For the Vircator #1, the maximum simulated average power was obtained for the point number 17. Fig. 4 displays the sampled points and its corresponding average power, obtained from the simulations. In the Fig. 4, both the grayscale and the radius of each point represent the average power obtained at the extraction window in the band from $5.3GHz$ to $7GHz$ ($1.9f_p$ to $2.5f_p$).

Notice that Fig. 4 presents the same behavior of Fig. 2. In fact, the two graphics display the same problem. The model was able of predict the couple of values $(V, r_c)$ maximizing the average power radiated.

The results predicted by the model and the results obtained by simulation for Vircators #2 and #3 are presented in Figs. 5 and 6. Notice that in both cases, the model was able to predict the values $(V, r_c)$ maximizing $\bar{P}$, even though it fails to predict the exact value of $\bar{P}$.

It can be concluded that the model and optimization procedure presented in Section III was able to find the optimal parameters in all three cases.
V. CONCLUSIONS

We presented a methodology to determine the optimal parameters to produce the higher amount of energy on a Vircator of axial extraction while its dominant frequency is a constrain.

The optimization approach presented was performed for fixed values of the drift-tube radius and the anode transparency. Despite this, the solution is useful for cases where the Vircator is available, and the anode transparency is chosen to ensure a lifetime.

At determined VC oscillation frequency, anode-cathode gap and the anode transparency do not alter the average power radiated. However, anode transparency variations can relocate the curve $I_d = I_c$ and so, the Vircator’s upper limit of energy production can be modified. Hence, optimal anode transparency is the higher possible.

The model can predict the energy produced by a Vircator of axial extraction close to the optimal parameters. But, it is not adequate to predict the average power for work points far from the maximum. The reason for this is that the VC charge model (Eq. (5)) was fitted to ease the solution of the optimization problem.

APPENDIX: PROOF THAT THE MAXIMUM ALWAYS FALLS ON THE CURVE $I_d = I_c$

It is necessary to prove that the maximum average power always drops on the curve $I_d = I_c$. To make this, it is enough to prove that $\bar{P}$ is an increasing function of $r_c$. Notice that $I_d = I_c$ defines the upper limit of $r_c$ when the other parameters are fixed.
Eq. (18) at fixed $V$, $r_{dt}$ and $\omega_p$ can be rewritten as

$$\bar{P}(r_c) = c_3 r_c^4 e^{-\left(1 - \frac{c_4}{\sqrt{1 + \ln \left(\frac{r_c}{r_{dt}}\right)}}\right)^2},$$  \hspace{1cm} (A1)

where $c_3$ and $c_4$ are positive constants.

Deriving Eq. (A1) respect to $r_c$ is obtained

$$\frac{d}{dr_c} \bar{P}(r_c) = 4c_3 e^{-\left(1 - \frac{c_4}{\sqrt{1 + \ln \left(\frac{r_c}{r_{dt}}\right)}}\right)^2} \left(r_c^2 \left(1 + \frac{c_4}{r_c}\right) - 2c_4\right)^2 \left(1 + \frac{\ln \left(\frac{r_c}{r_{dt}}\right)}{r_c}\right)^2.$$  \hspace{1cm} (A2)

Due that $r_{dt} > r_c$, Eq. (A2) is always positive. Hence, the average power is a monotonic increasing function of $r_c$. Because of this, maximum $\bar{P}$ at given $V$ is reached at the maximum $r_c$ which is defined by the curve $I_{b} = I_{c}$.